Operator Spreading and Simulation of $\mathbb{Z}_3$ Parafermions

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Abstract

$\mathbb{Z}_q$ parafermions generalize Majorana fermions, the $q = 2$ case, to operators and excitations with $\mathbb{Z}_q$ exchange algebra. In 1D $\mathbb{Z}_q$-symmetric chains, such as chiral clock models or the achiral Potts models, parafermions are domain wall excitations. Now, there is promise for realizing parafermions on quantum devices [8-13], and probing their dynamics. A particularly promising route is to study dynamics near the dual-unitary points, where exact solutions exist [4]. We study operator spreading of parafermionic operators via fidelity metrics and OTOCs with a particular focus on the $\mathbb{Z}_3$ model. Finally, we find an efficient encoding of the $\mathbb{Z}_3$ chiral clock model unitaries, and outline how dynamics can be simulated.

Selected Previous Works

My work has centered on non-equilibrium condensed matter systems, including fermions, bosons, and spins, primarily focusing on using drives and dissipation to stabilize desirable quantum states and phases [14-18]. I have used techniques including Keldysh field theory for Lindblad systems, exact numerics, and tensor networks.

$\mathbb{Z}_q$ Chiral Clock Models

The 1D transverse-field Ising model can be generalized to on-site dimension q and chiral couplings ϕ, θ

$$H = -J\sum_{n=1}^{N-1}\left[e^{i\phi}\sigma_n\sigma_{n+1}^\dagger + h.c.\right] - f\sum_{n=1}^{N}\left[e^{i\theta}\tau_n + h.c.\right]$$

The $q = 2$ model is not chiral as $\sigma = \sigma^+ = Z$, but for $q \geq 3$ the model can be chiral, with clock operator σ, shift operator τ, and $\omega = e^{2\pi i/q}$. A $q{\times}q$ matrix form is

$$\sigma = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots \end{pmatrix}, \quad \tau = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \end{pmatrix}$$

which obey the clock algebra $\sigma\tau = \omega\tau\sigma$ and $\sigma^q = \tau^q = 1$.

$\mathbb{Z}_q$ Parafermions

This model may be rewritten via the Fradkin-Kadanoff transform into parafermions χ, ψ with $\eta = e^{i\pi(q-1)/q}$ [1]

$$\chi_n = \left(\prod_{m=1}^{n-1}\tau_m\right)\sigma_n, \qquad \psi_n = \eta\,\chi_n\tau_n$$

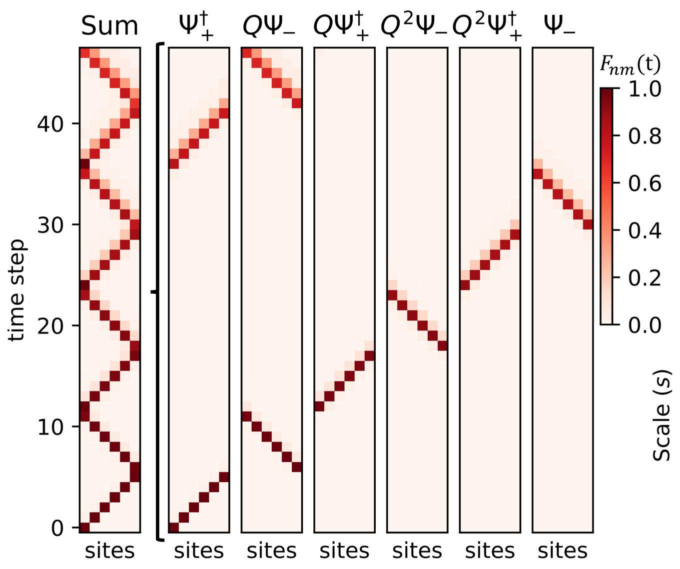
and $\chi_n\psi_n = \omega\psi_n\chi_n$, $\chi_n\psi_m = \omega\psi_m\chi_n$, $\chi_n\chi_m = \omega\chi_m\chi_n$, $\psi_n\psi_m = \omega\psi_m\psi_n$ for $n < m$, $\chi_n^q = \psi_n^q = 1$. H is then

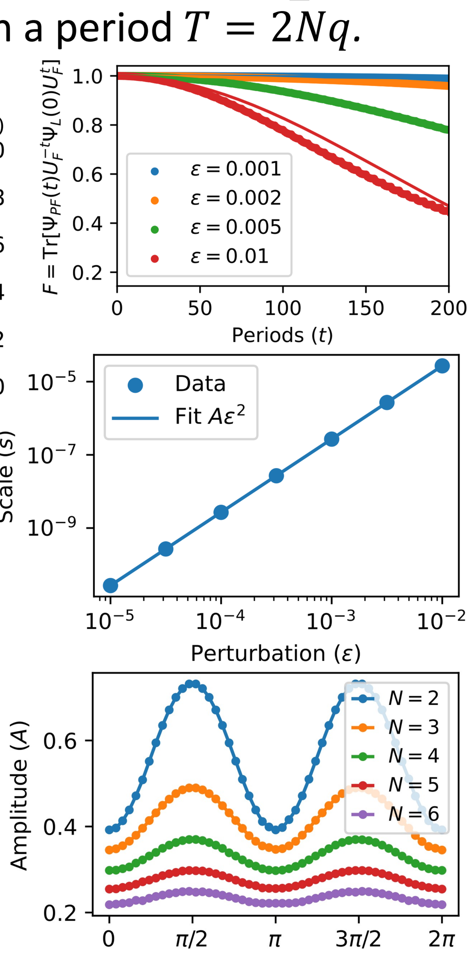
$$H = -J\sum_{n=1}^{N-1}\left[\eta e^{i\phi}\chi_{n+1}^\dagger\psi_n + h.c.\right] - f\sum_{n=1}^{N}\left[\frac{e^{i\theta}}{\eta}\chi_n^\dagger\psi_n + h.c.\right]$$

The algebra suggests that braiding parafermions zero mode defects in suitable networks of wires or in 2D enables topological quantum computation [2]. Braiding $\mathbb{Z}_q$ parafermions allows a richer set of unitaries than $\mathbb{Z}_2$ Majorana fermions and can generate entanglement [3].

Coherent Dephasing in the $\mathbb{Z}_3$ Model

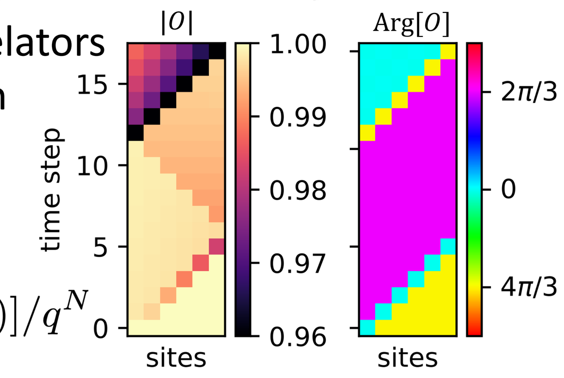
For $f\delta t, J\delta t$ near the dual unitary point, motion is nearly ballistic with some spreading. At the boundary, $\Psi_\pm$ swap and change $\mathbb{Z}_q$ sectors, resulting in a period $T = 2Nq$.





The decay is Gaussian, $\exp(-st^2)$, and with the decay rate set by the size of the perturbation ϵ from dual unitarity. Crucially, the scaling is $A\epsilon^2$ which corresponds to coherent dephasing [6]. The exponent is the same for different perturbations in the $f-J$ plane, but the amplitude varies with direction d as $A_0 + A'\sin^2(d)$. I expect that in large systems $A' \to 0$, and is only finite because f acts N times to J's $N-1$.

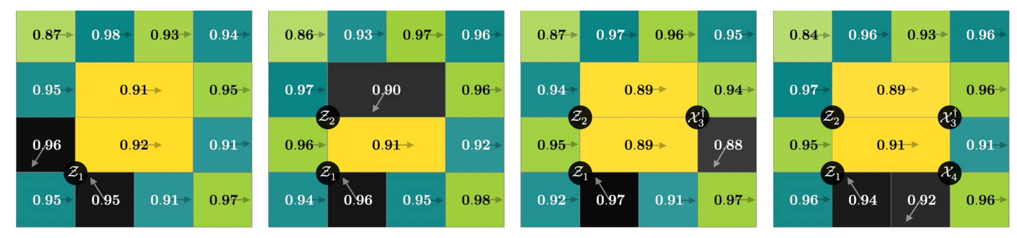
Now, out of time-order correlators provide more information on the operator spreading [7], here diagnostics for braiding phase. Up to site indices it is $O(t) = \mathrm{Tr}[\Psi_{\mathrm{PF}}^\dagger\Psi_0^\dagger(t)\Psi_{\mathrm{PF}}\Psi_0(t)]/q^N$



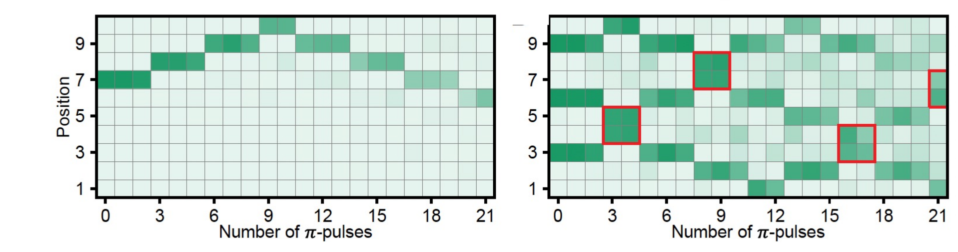
Existing Quantum Simulations of $\mathbb{Z}_q$

Here we highlight previous works focusing on $q = 3$. Some work has focused on entangled state preparation [8], and realizing qudit lattice gauge theories [9]. We now select four examples to focus on in more detail.

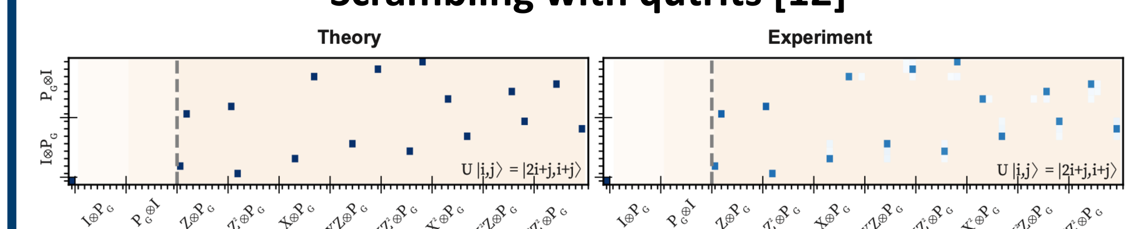
Qutrit toric code [10]



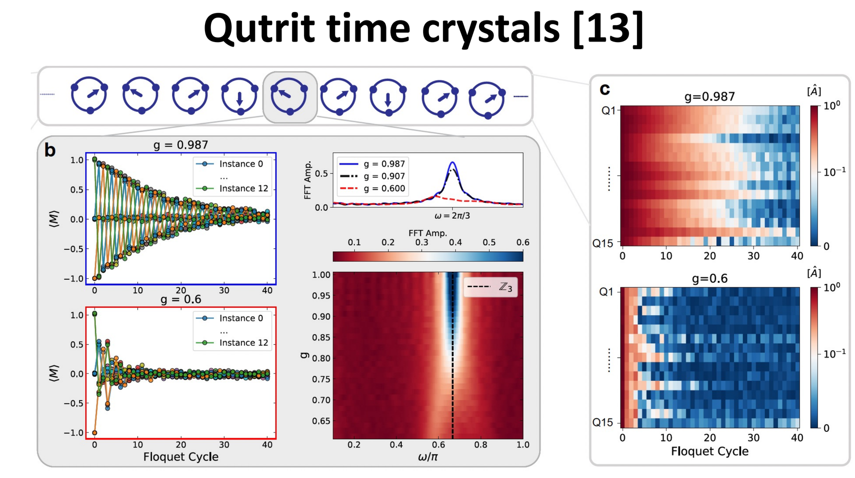
Quantum cellular automata ($\mathbb{Z}_2$) [11]



Scrambling with qutrits [12]

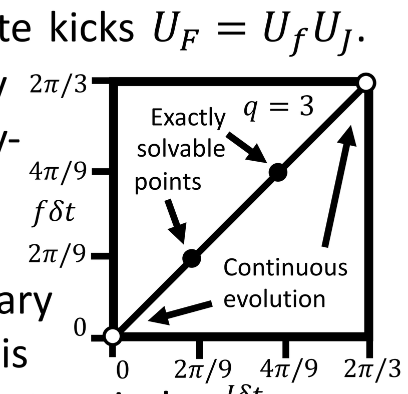


Qutrit time crystals [13]



Floquet Dual-Unitary Points

Now, let us consider the kicked $\mathbb{Z}_q$ model with finite δt where the system evolves in discrete kicks $U_F = U_f U_J$. Curiously, this model has two exactly solvable points [4], where Temperley-Lieb algebra / transfer matrices fully specifies dynamics and correlations. Physically, these points are dual unitary where translation by one lattice site is the same as translating by one Floquet period.



Ballistically Propagating Parafermions

Next, Claeys and Lamacraft showed that parafermions Ψ propagate ballistically at the exactly solvable point [5]. The left/right movers obey

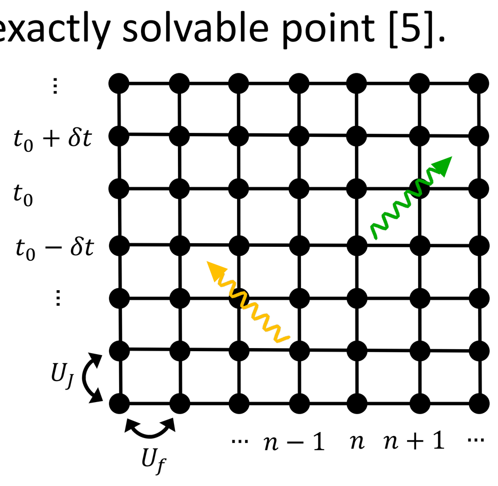
$\Psi_\pm(n \pm 1) = U_F^\dagger\Psi_\pm(n)U_F$ where $\Psi_\pm = D^+\bar{\Psi}_\pm D$ for a diagonal phase matrix D

$$\bar{\Psi}_+(n) = \left(\prod_{m<n}\sigma_m\tau_m\right)\tau_n$$

$$\bar{\Psi}_-(n) = \left(\prod_{m<n}\sigma_m\tau_m\right)\sigma_n$$

which for $n > m$ satisfy

$$\Psi_\pm(n)\Psi_\pm(m) = \omega^{\pm 1}\Psi_\pm(m)\Psi_\pm(n)$$



Perturbations Away From Dual Unitarity

Leaving a dual unitary point one expects an operator fidelity like $F(t,\epsilon) = \mathrm{Tr}[\Psi_0^+(t)\Psi_\epsilon(t)]/q^N$ to fall off with time where Ψ_0 and Ψ_ϵ are the operators evolved at the dual unitary point and perturbed respectively. More specifically, a coherent perturbation whose correlations remain non-decaying over the observed time window, leads to a Gaussian fall-off $\exp(-s(\epsilon)t^2)$, with exponent $s(\epsilon) = A\epsilon^2 + O(\epsilon^3)$ to leading order [6].

Efficient Encoding Into Gates

Discrete unitary gates are possible on digital quantum devices. These gates are often restricted to one- and two-qubit unitaries, typically with an additional spatial-locality constraint. Quantum simulation of condensed matter models must fit within these constraints, and the fact that two-qubit gates have sizeable error rates means gate counts should be minimized.

Unitary Decomposition

The Trotter decomposition for smooth time-evolution is

$$e^{(-iH_1 - iH_2 - iH_3)\delta t} = \underbrace{e^{-iH_1\delta t}}_{U_{J\text{-odd}}}\underbrace{e^{-iH_2\delta t}}_{U_{J\text{-even}}}\underbrace{e^{-iH_3\delta t}}_{U_f} + O(\delta t^2)$$

where we consider

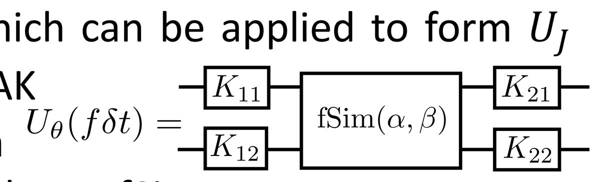
$$U_{J\text{-odd}} = \exp\Big[iJ\delta t\sum_{n=1,3,5,\ldots} e^{i\phi}\sigma_n\sigma_{n+1}^\dagger + e^{-i\phi}\sigma_n^\dagger\sigma_{n+1}\Big]$$

$$U_f = \exp\Big[if\delta t\sum_{n=1,2,3,\ldots} e^{i\theta}\tau_n + e^{-i\theta}\tau_n^\dagger\Big]$$

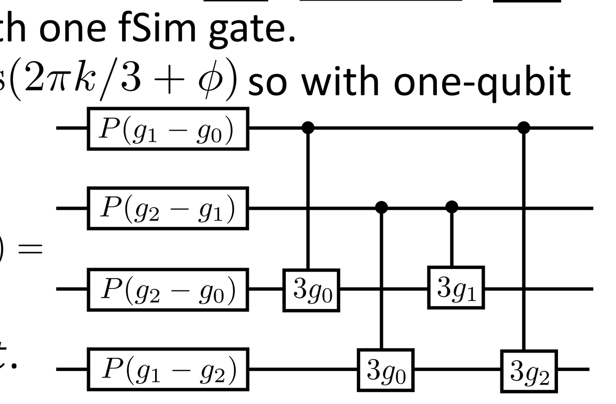
For the kicked model, $U_F = U_{J\mathrm{odd}}U_{J\mathrm{even}}U_f$, exactly.

Unitary Encoding

We encode one qutrit into two qubits using Gray coding where $|0\rangle_L = |00\rangle$, $|1\rangle_L = |10\rangle$, $|2\rangle_L = |11\rangle$, and $|01\rangle$ is a leakage state. We will now be interested in the local operators U_θ and U_ϕ which can be applied to form U_J and U_f. Now using the KAK decomposition, there is a representation for U_θ with one fSim gate.



Next, let $g_k = -2J\delta t\cos(2\pi k/3 + \phi)$ so with one-qubit gate $P(z) = \mathrm{diag}(1, e^{iz})$ and the four entangling CPhase gates we can express U_ϕ exactly for all J, ϕ, and δt.



Quantum Simulation Proposal

Sampling Shot by Shot

Consider fidelity $F_{nm}(t) = \mathrm{Tr}[\Psi_{\mathrm{PF}}^+(n)U_F^{-t}\Psi_0(m)U_F^t]/q^N$ this can be evaluated by fixing a complete basis $|\psi_i\rangle$ with $\Psi_{\mathrm{PF}}(n)|\psi_i\rangle = c_i|\psi_i\rangle$ so that

$$F_{nm}(t) = \frac{1}{q^N}\sum_i c_i^*\langle\psi_i|U_F^{-t}\Psi_0 U_F^t|\psi_i\rangle$$

From this we can now uniformly sample ψ_i with shots s.

$$F_{nm}(t) \approx \frac{1}{N_{\mathrm{shots}}}\sum_{\mathrm{shots}\ s} c_s^*\langle\psi_s(t)|\Psi_0|\psi_s(t)\rangle$$

The steps are (1) prepare random initial state $|\psi_s\rangle$, (2) evolve $|\psi_s(t)\rangle = U_F^t|\psi_s\rangle$ with U_F known, (3) measure operator expectation values in Ψ_0's measurement basis, (4) repeat, (5) compute F, taking leakage into account.

Next Steps

1. Tensor network numeric calculations for large systems
2. Extend from $\mathbb{Z}_3$ to $\mathbb{Z}_4$ – when does $\mathbb{Z}_4 \sim \mathbb{Z}_2 \times \mathbb{Z}_2$ matter?
3. Run simulation of dynamics on a quantum device

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