Electronic localization-delocalization transition in a "zero-quadratic" model of quantum Hall states

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Motivation

- In a magnetic field, electrons occupy states known as Landau Levels (LLs)
  - These states exhibit interesting behaviors such as quantized transport properties
- When interactions with the lattice become significant (e.g. an insulator) electrons occupy non-LL states whose behavior differs from that of LLs
  - Can probe electronic properties that are obscured by the high symmetry of LLs
  - Here: a model where lattice effects eliminate quadratic momentum dependence

$$H \propto p^2 \qquad \mapsto \qquad H \propto p^4$$

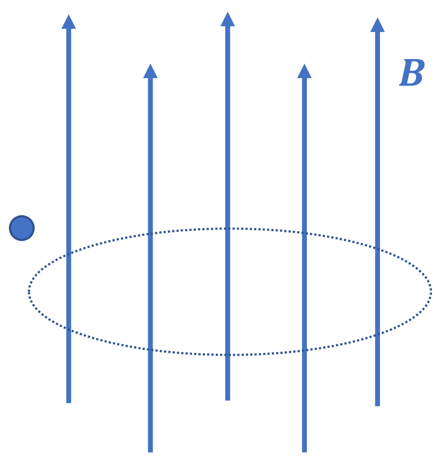
- By understanding the behavior of models we can place, frame, and inspire the engineering of materials with novel properties and applications

Lorentz Force Law

- Charged particles experience forces in E and B fields

$$\vec{F} = m\vec{a} \qquad \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

- Cyclotron motion



Landau Levels

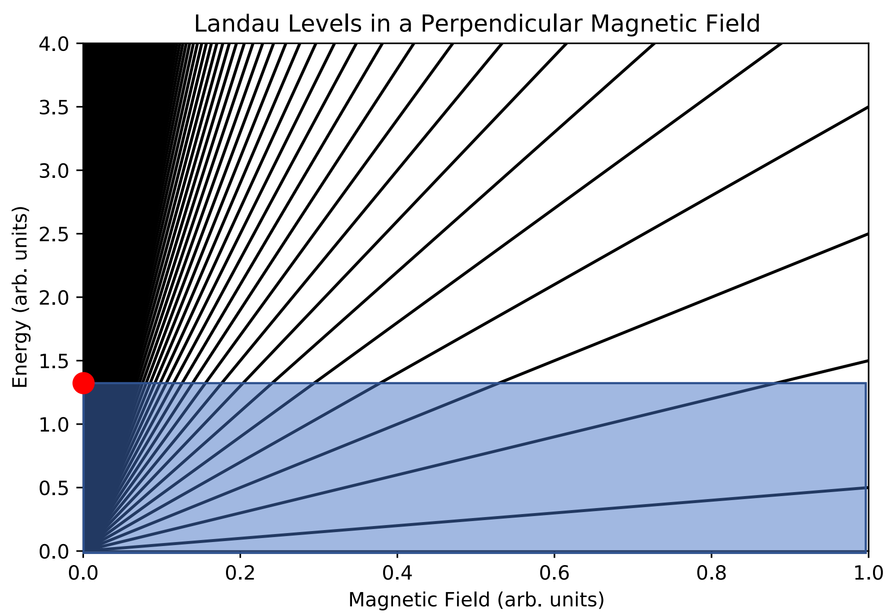
- Select vector potential $\vec{A} = (0, Bx, 0)$, then:

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right)$$

- Quantized angular momentum $L = n\hbar$
    - Equally spaced energy levels

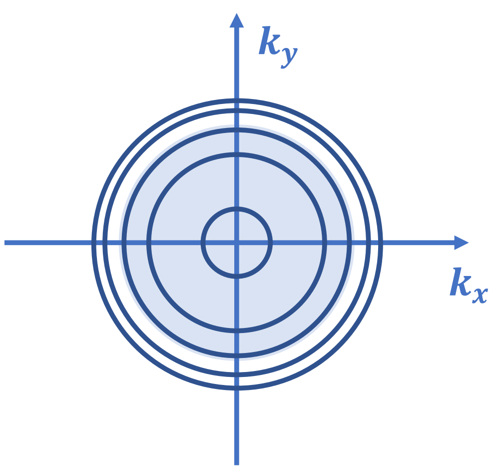
$$E_n = \hbar\frac{eB}{m}\left(n + \tfrac{1}{2}\right)$$

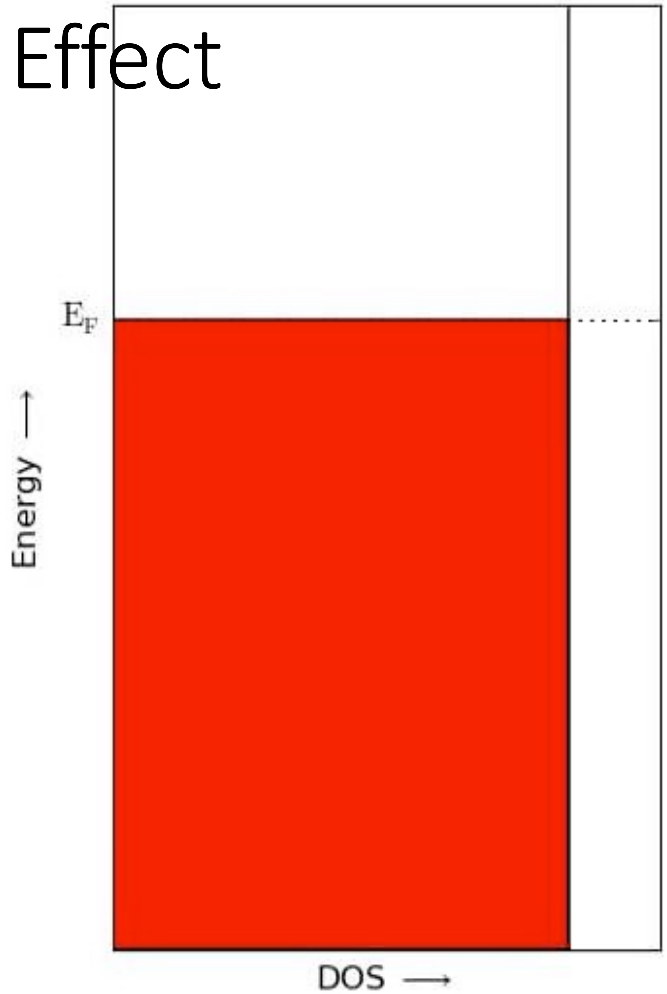
- Linear dependence on B

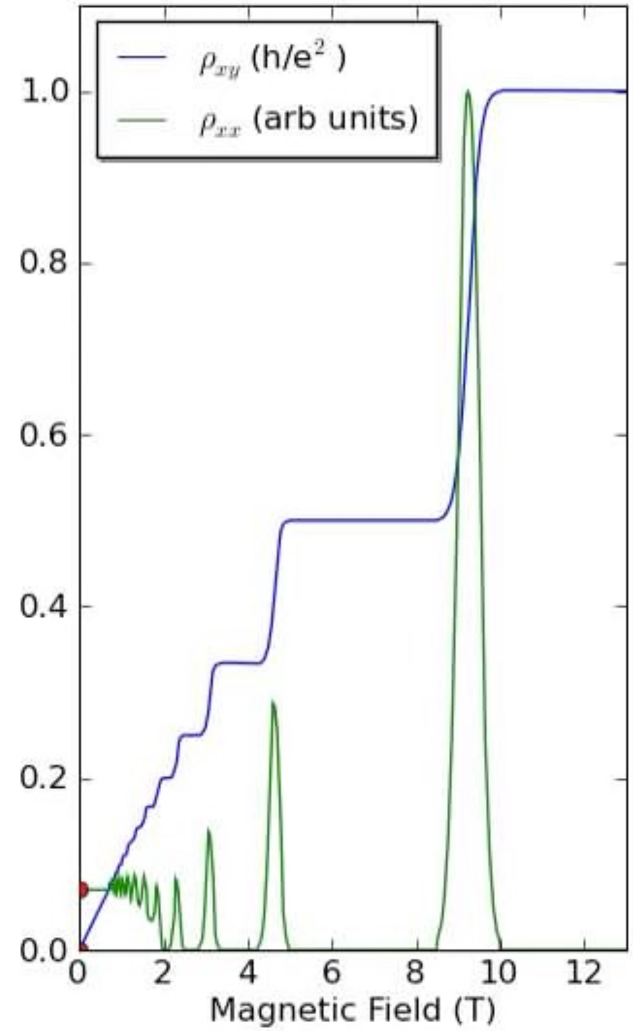


Quantum Hall Effect

Quantized transverse resistance dependent on number of LLs within Fermi Surface





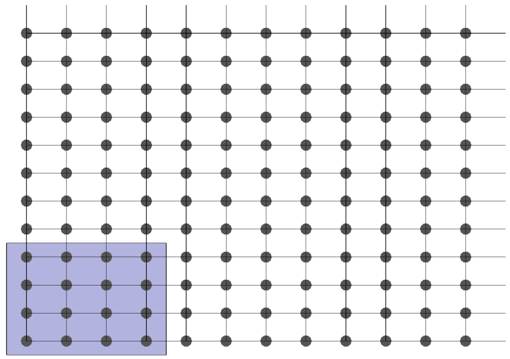


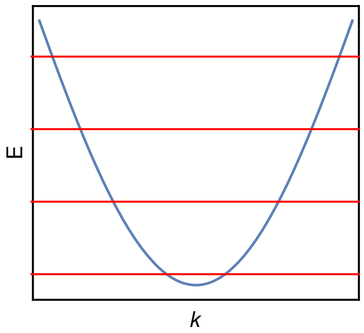
Von Klitzing, *et al*, PRL **45**, 494 (1980).
Thouless, *et al*, PRL **49**, 405 (1982).

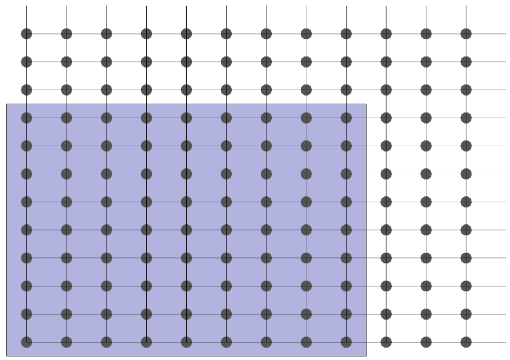
Video from Wikimedia commons

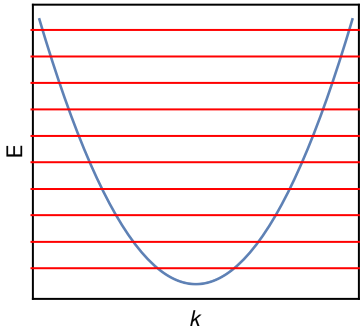
Tight-Binding Models

- We saw a metal-insulator transition
  - Only the occupied LLs are conducting
- Analysis before only considers behavior in an electron gas (metal)
- Tight-binding models consider electrons attached to orbitals of atoms in a solid (insulator)
  - Study behavior across the transition

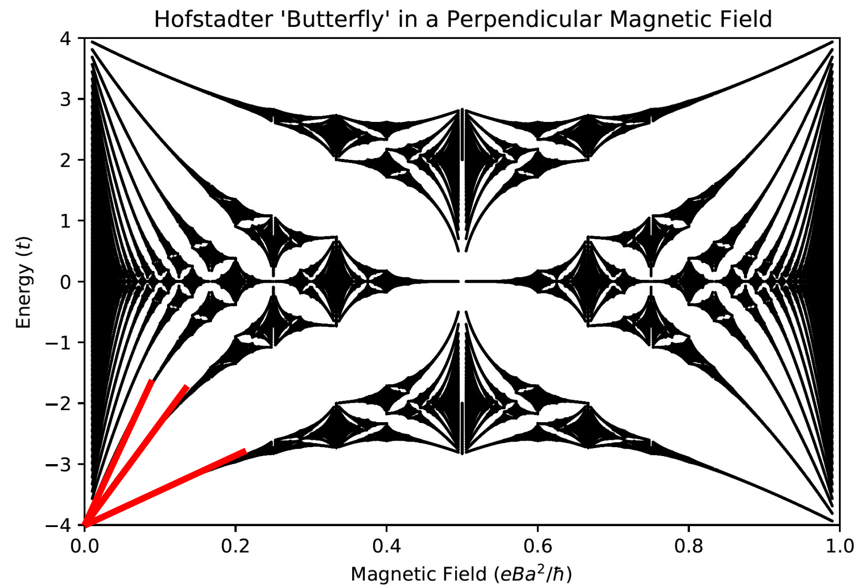








Hofstadter Model



- Lattice with nearest-neighbor hoppings
- Select vector potential $\vec{A} = (0, Bx, 0)$
- Quadratic in momenta to lowest order

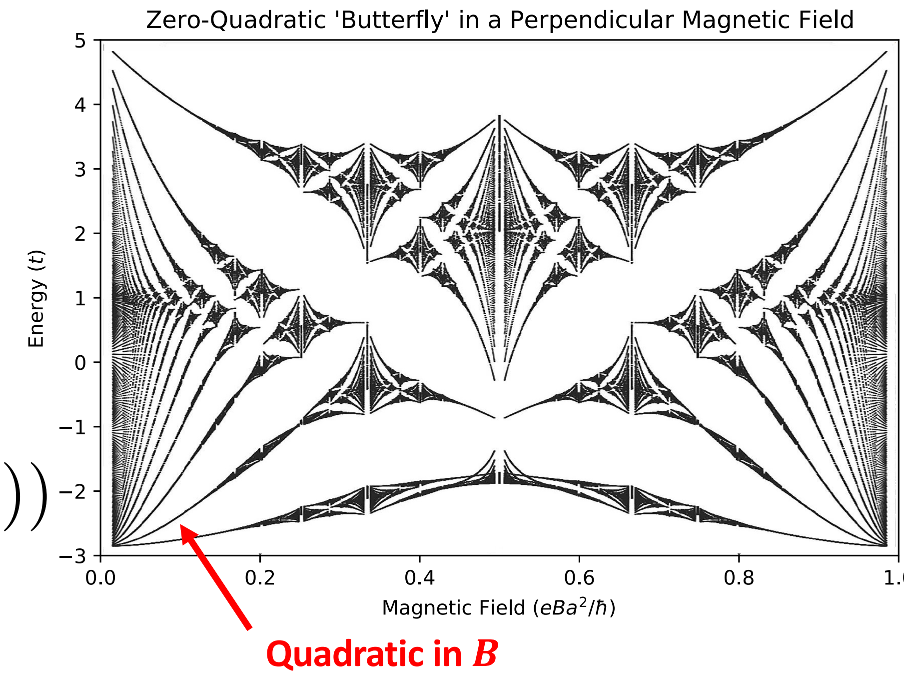
$$H = 2t(\cos(K_x) + \cos(K_y))$$
$$\approx -4t + (K_x^2 + K_y^2)$$

- Energy is linear in magnetic field (if the magnetic field is small)

$$E_n = -4t + 2t(eBa^2/\hbar)(n + \tfrac{1}{2})$$

Hofstadter, PRB **14**, 2239 (1976).

Zero-Quadratic Model



- Next-nearest-neighbor hoppings too
- Hamiltonian is quartic in momenta

$$H = 2t(\cos(K_x) + \cos(K_y)) - \tfrac{1}{2}t(\cos(2K_x) + \cos(2K_y)) \approx -3t + \tfrac{1}{4}t(K_x^4 + K_y^4)$$

- Energy is quadratic in magnetic field (if the magnetic field is small)

$$E_n = -3t + \tfrac{1}{16}t(eBa^2/\hbar)^2[12(n + \tfrac{1}{2})^2 + 3]$$

How to Study This Phase Transition?

- Metal insulator phase transition
- Electrons are exponentially localized around position r_0

$$\langle \psi(r_0) | \psi(r) \rangle \propto e^{-|r-r_0|/\xi}$$

- 'Localization length' parameter ξ determines how metallic/insulating
- Calculate ξ as a function of B, or as a function of Fermi energy
  - Transfer matrices
  - Lyapunov exponents
- Describe how ξ varies around phase transition

Pichard and Sarma, JPC **14**, 127 (1981)
MacKinnon and Kramer, PRL **47**, 1546 (1981)

Localization Length Dependence on Energy

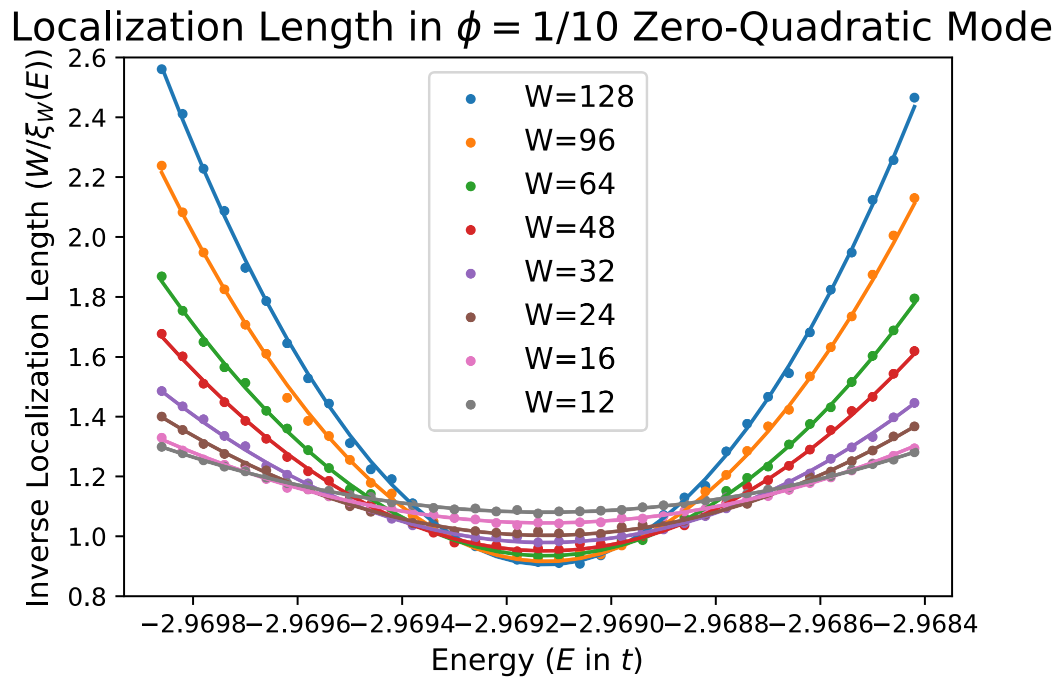
- Infinite sample:

$$\xi_\infty(E) \propto |E - E_c|^{-\nu}$$

- Finite sample:

$$\frac{W}{\xi_W(E)} \approx a_0 + a_2 (E - E_c)^2 W^{2/\nu}$$

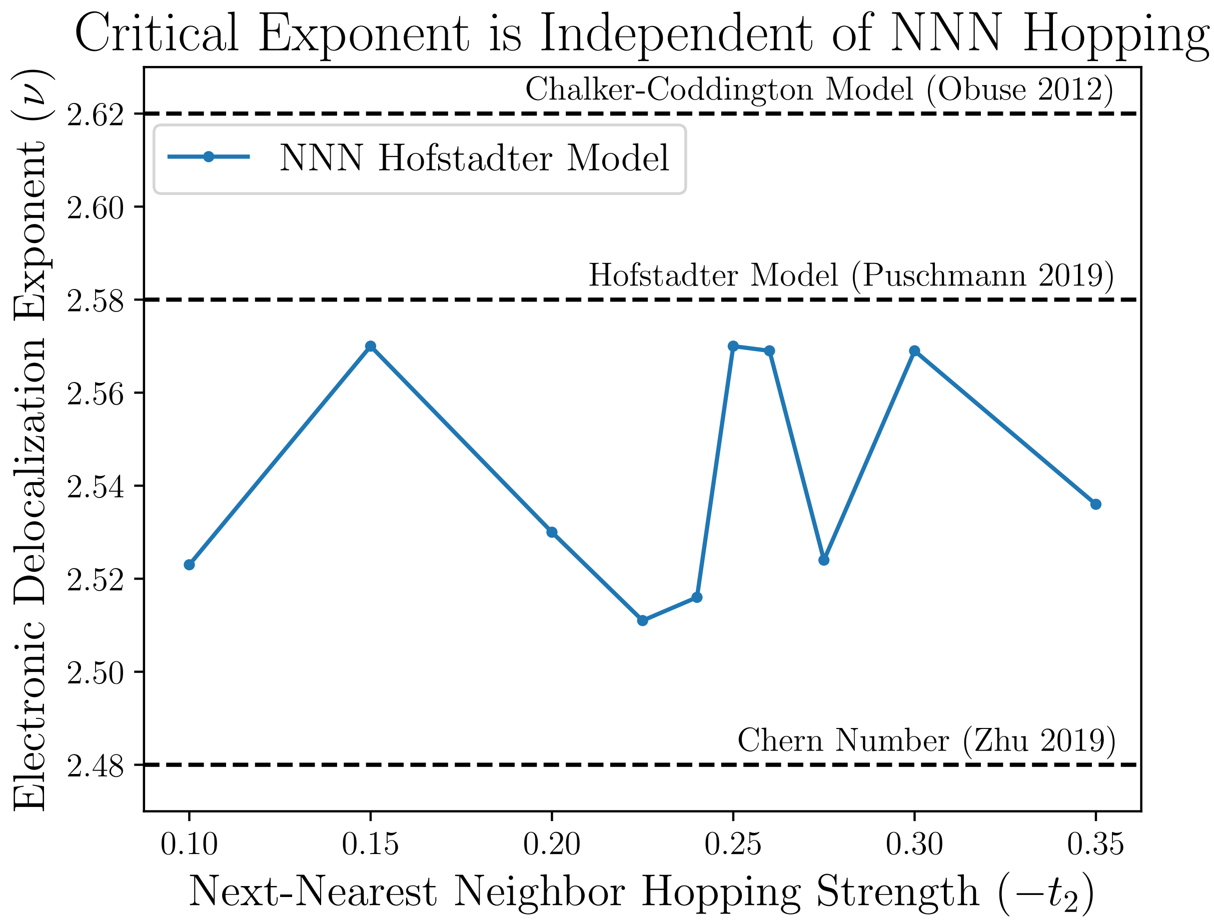
- Fit a_0, a_2, ν from data



Results

- Critical exponent ν of the phase transition is independent of NNN hopping strength
- Exponent is in good agreement with Hofstadter model and continuum model (Chern number)

Obuse, *et al*, PRL **109**, 206804 (2012)
Puschmann, *et al*, PRB **99**, 121301 (2019)
Zhu, *et al*, PRB **99**, 024205 (2019)



Conclusion

- Electrons exhibit a wide variety of behaviors, some of which are leveraged to enable device technologies
- Here we consider electrons in a magnetic field subject to lattice effects
  - H, lattice effects eliminate the quadratic dependence of the Hamiltonian
- These zero quadratic states can be used in fractional quantum Hall physics as basis states for quasipartical “droplets”, which may be braided to encode quantum information
  - These states enable the geometrical degree of freedom of the droplets to be leveraged, and which could lead to greater control of the encoded information
- We find that the phase transitions between zero quadratic states are indistinguishable from those of quadratic states

Acknowledgements

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Form of the K operators

In the presence of the magnetic field, the naïve translation operators do not transform correctly under gauge transformations [2], and we must accompany translations by compensatory gauge transformations. The translation operators with the appropriate transformation properties are a sum over lattice sites

$$T_a = \sum_m e^{i\theta_a(m)} c^\dagger_{m+e_a} c_m. \quad (1)$$

where the phases $e^{i\theta_a(m)}$ satisfy

$$\theta_1(m) + \theta_2(m+e_1) - \theta_1(m+e_2) - \theta_2(m) = \phi. \quad (2)$$

The components of T do not commute, but satisfy

$$T_x T_y = \exp(i\phi) T_y T_x. \quad (3)$$

The lattice translation operators T_a are unitary, so we can write them in terms of Hermitian generators $T_a = \exp(iK_a)$. The K_a are the lattice analogues of the covariant momentum operators π_a, and we will sometimes call them momenta for brevity. These operators have the commutator

$$[K_x, K_y] = \phi. \quad (4)$$

Hall Effect

- Two dimensional sample
- With electric and magnetic fields

$$\vec{E} = E_0 \hat{x} \qquad \vec{B} = B_0 \hat{z}$$

- Classical Limit (small B, large T)

$$\rho_{xy} = \frac{B}{ne}$$

- Quantum Limit (large B, small T)

$$\rho_{xy} = \frac{h}{\nu e^2}$$

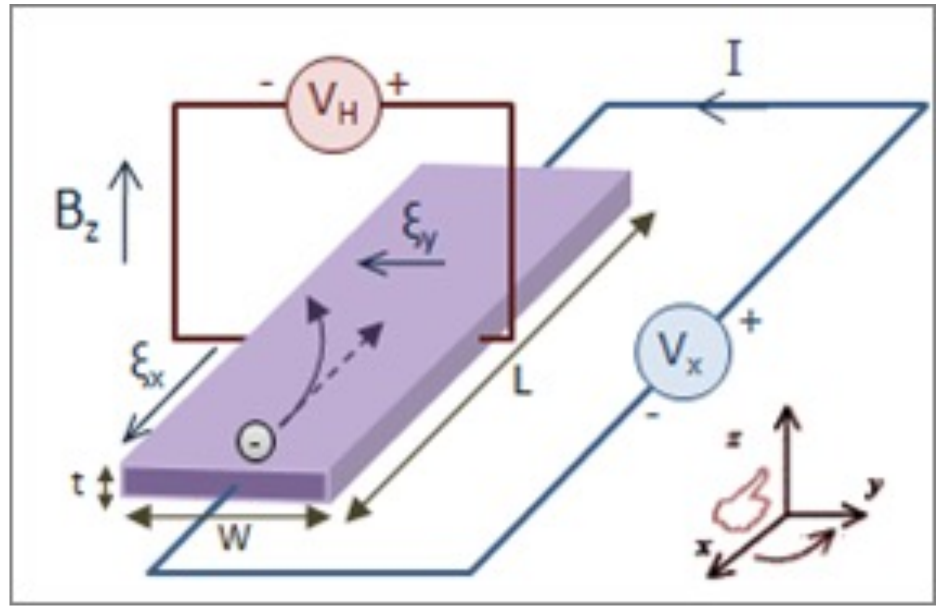

Image from Wikimedia Commons

Why $t_2 = -1/4$

As in Section II A, we write this in terms of the Hermitian generators of lattice translations,

$$H_{\mathrm{TB}} = -2\left(\cos(K_x) + \cos(K_y)\right) - 2t_2\left(\cos(2K_x) + \cos(2K_y)\right) \tag{19}$$

Replacing the cosine terms by their Taylor expansion, the terms lowest-order in the momenta are

$$H_{\mathrm{TB}} = -4 - 4t_2 + (1 + 4t_2)\left(K_x^2 + K_y^2\right) - \left(\frac{1}{12} + \frac{4}{3}t_2\right)\left(K_x^4 + K_y^4\right) + \dots \tag{20}$$

$1 + 4t_2 = 0$
if $t_2 = -1/4$

Transfer Matrices

- Iteratively solve Schrödinger Equation

$$\begin{pmatrix} \ell+2 \\ \ell+1 \\ \ell \\ \ell-1 \end{pmatrix} = A_\ell \begin{pmatrix} \ell+1 \\ \ell \\ \ell-1 \\ \ell-2 \end{pmatrix}$$

- Use the matrix

Rearranging, we find that

$$A_\ell = \begin{pmatrix} (t_1/t_2)1 & (\bar{H}_\ell - E1)/t_2 & (t_1/t_2)1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad (74)$$

Where the intra-layer Hamiltonian is, where $m = (\ell, w)$,

$$\bar{H}_\ell = \big(\sum_w \epsilon_{\ell,w} a^\dagger_{\ell,w} a_{\ell,w}\big) - t_1(T_{y,\ell} + T^\dagger_{y,\ell}) - t_2(T^2_{y,\ell} + T^{\dagger 2}_{y,\ell}), \qquad (72)$$

with the intra-layer translation operators

$$T_{y,\ell} = \sum_w e^{-i(2\pi\phi/\phi_0)\ell} a^\dagger_{\ell,w+1} a_{\ell,w}. \qquad (73)$$

Localization Length

- QR Decomposition (eigenvalues)

Thus, the component of the wavefunction in the last layer is given in terms of the wavefunction in the first layer as [57–60]

$$|L\rangle = \left(\prod_{\ell=1}^{L} A_\ell\right)|1\rangle = \left(Q_L \prod_{\ell=1}^{L} R_\ell\right)|1\rangle, \qquad (75)$$

- Localization length from most conductive state (via Lyapunov Exponent)

Now, the localization-length is related to the the transfer matrix through the set of Lyapunov exponents γ_w [57–60]:

$$\xi = \frac{1}{\min_w |\gamma_w|}. \qquad (76)$$

Where Lyapunov exponents are given by the sum of the eigenvalues of A_ℓ (diagonal elements of R_ℓ) scaled by the logarithm,

$$\gamma_w = \frac{1}{L}\sum_{\ell=1}^{L} \ln |R_\ell^{(w,w)}|. \qquad (77)$$