Example: A thin film that gets thicker with time

Consider a thin film of soap suspended inside a ring, whose thickness is given by $d(t) = d_0 t$.

(a) If the index of refraction of the film is n, and light is normally incident with vacuum wavelength λ, at what times does the reflected light experience interference maxima and minima? Assume that at each interface, half of the light is reflected and half is transmitted.

The wavelength in the material is $\lambda' = \lambda/n$, and the interference maxima[1] will be when $2d = k\lambda'$ for $k \in \mathbb{Z}$, which is when $t = k\lambda'/2d_0 = k\lambda/2nd_0$. The interference minima[2] will be when $2d = (k + \frac{1}{2})\lambda'$, which is when $t = (k + \frac{1}{2})\lambda'/2d_0 = (k + \frac{1}{2})\lambda/2nd_0$.

(b) Assuming the initial electric field magnitude is E, what is the magnitude of the electric field of the reflected light at the interference maxima? What is the magnitude of the electric field of the reflected light at the interference minima?

At the interference maxima:

$$E_{\text{ref}} = \frac{E}{2} + \frac{E}{2^3} + \frac{E}{2^5} + \frac{E}{2^7} + \dots \tag{1}$$

where the powers of 2 correspond to the number of interfaces the light encounters.

Now recall the geometric series formula:

$$\sum_{i=1}^{\infty} \frac{1}{p^i} = \frac{1}{p-1} \tag{2}$$

if $p > 1$.

Here:

$$E_{\text{ref}} = \sum_{i=1}^{\infty} \frac{E}{2^{2n-1}} = 2E \sum_{i=1}^{\infty} \frac{1}{4^n} = \frac{2}{3}E \tag{3}$$

At the interference minima:

$$E_{\text{ref}} = \frac{E}{2} - \frac{E}{2^3} + \frac{E}{2^5} - \frac{E}{2^7} + \frac{E}{2^9} - \frac{E}{2^{11}} + \dots \tag{4}$$

where the minus signs correspond to destructive interference. Not all terms are destructive because destructive interference corresponds to $2d = (k + \frac{1}{2})\lambda'$, so with two journeys from top to bottom and back inside the film, $4d = (2k+1)\lambda'$ which corresponds to constructive interference.

We have:

$$E_{\text{ref}} = \frac{E}{2} - \sum_{i=1}^{\infty} \frac{E}{2^{4n-1}} + \sum_{i=1}^{\infty} \frac{E}{2^{4n+1}} \tag{5}$$

$$= \frac{E}{2} - 2E \sum_{i=1}^{\infty} \frac{1}{16^n} + \frac{E}{2} \sum_{i=1}^{\infty} \frac{1}{16^n} \tag{6}$$

$$= \frac{E}{2} - 2E\frac{1}{15} + \frac{E}{2}\frac{1}{15} \tag{7}$$

$$= \frac{2}{5}E \tag{8}$$

[1] first order
[2] first order