# Keldysh Input-Output Theory:
## Application to Cavity Quantum Materials


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with Ben Kass and Martin Claassen


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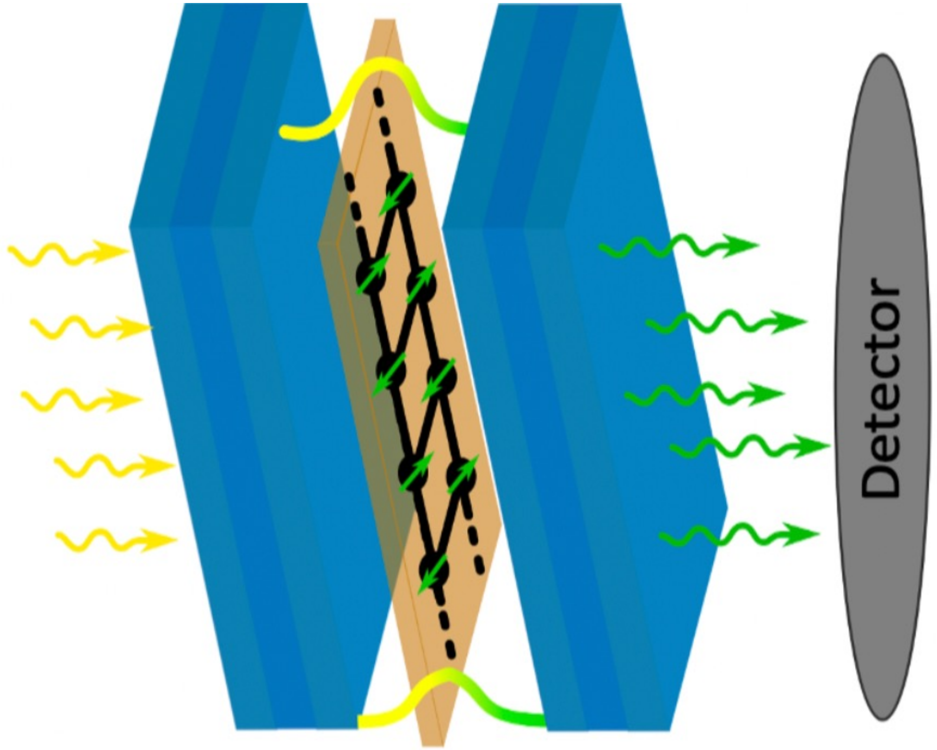
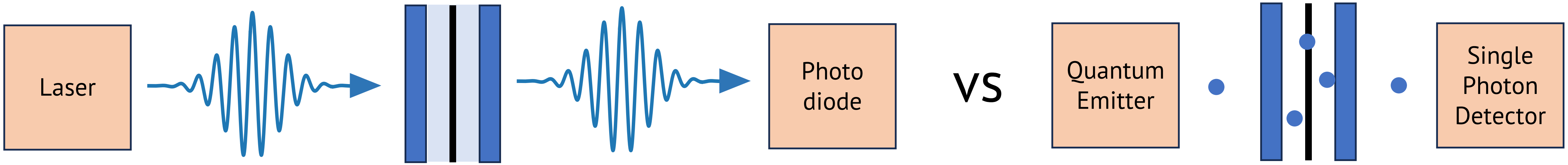


# Quantum Photon Spectroscopy of Cavity Materials

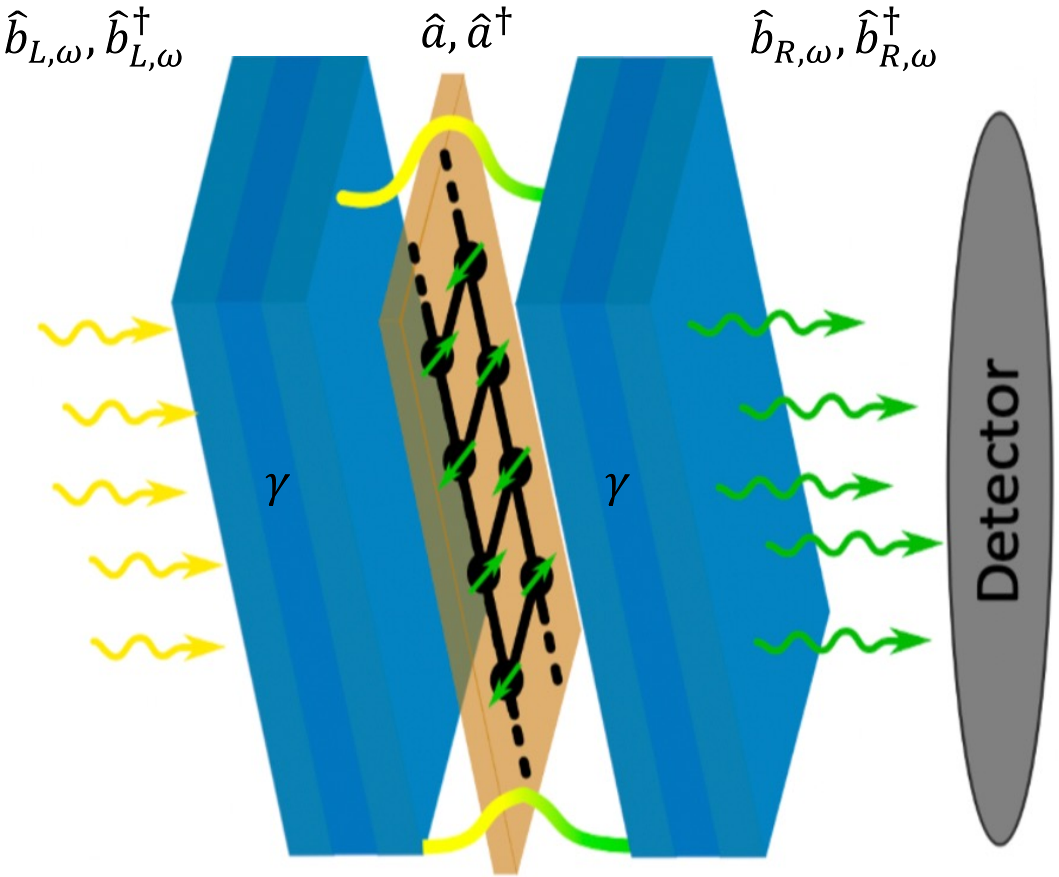




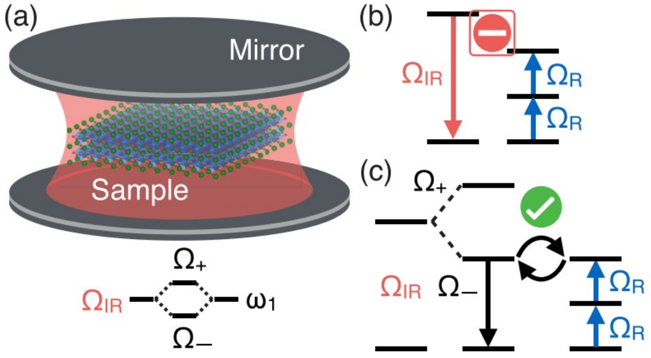

- Quantum light spectroscopy is a different probe than classical light
  - Few photon input and output states, possibly with entanglement
  - Don't destroy the state and can probe subtle quantum superpositions of states
  - In a cavity we can have multiple scattering processes
    - Note: neutron and x-ray scattering just have one scattering between source and detector
- Additionally we can convert from the correlation functions of the material in the cavity to the experimentally observable output fields
- Here: Keldysh input-output theory as a way to calculate input-output correlation functions for cavity quantum materials with quantum light

# Cavity Quantum Materials

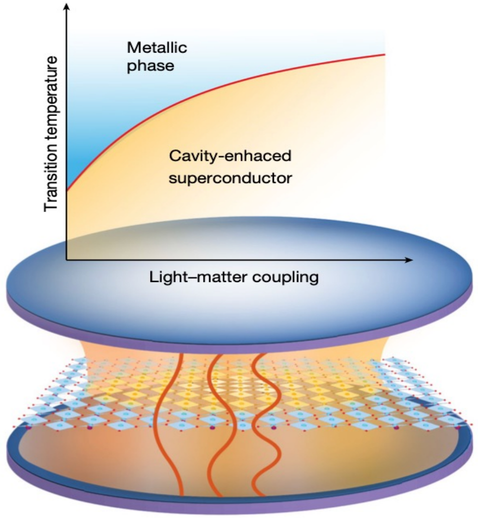
- Many recent proposals on cavity quantum materials for exotic physics
- How to probe states?



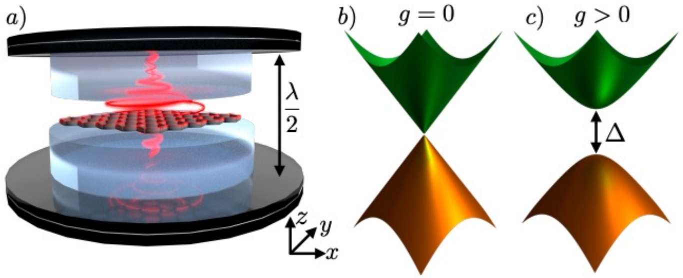

Nonlinear phononics
PRR 3, L032046 (2019)




Metal-superconductor transition
Nature 606, 41 (2022)




Cavity induced Chern insulators
PRB 99, 235156 (2019)



# Input-Output Theory



- Standard quantum optical input-output relations
  - Gardiner and Collett PRA 31, 3761 (1985)
  - Solve equations of motion for bath field operators: empty cavity with leakiness γ

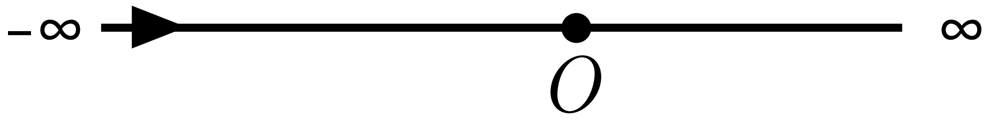
$$\hat{b}_{\text{out}}(t) = \hat{b}_{\text{in}}(t) + \gamma \hat{a}(t)$$

- Generalize to extended systems in a cavity (vs few quantum emitters)
- What is missing that we gain with field theory?
  - Rigorous perturbation theory in powers of many-body operators
    - E.g. $g^{(1)}$ and $g^{(2)}$ functionals in terms of pair gain/loss operators coupling the material to cavity modes
  - Effective descriptions of collective modes
    - E.g. superconducting modes from fermions
    - E.g. magnons from spins

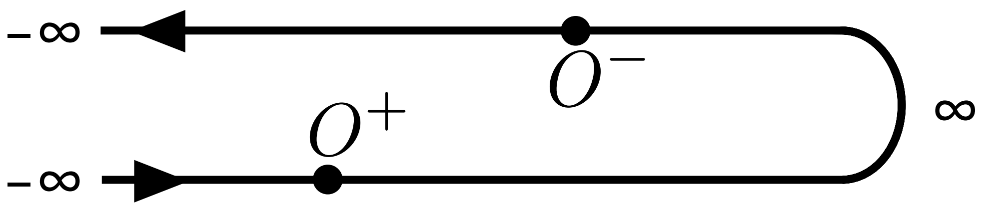
# Contours and Correlators



- Familiar equilibrium contour

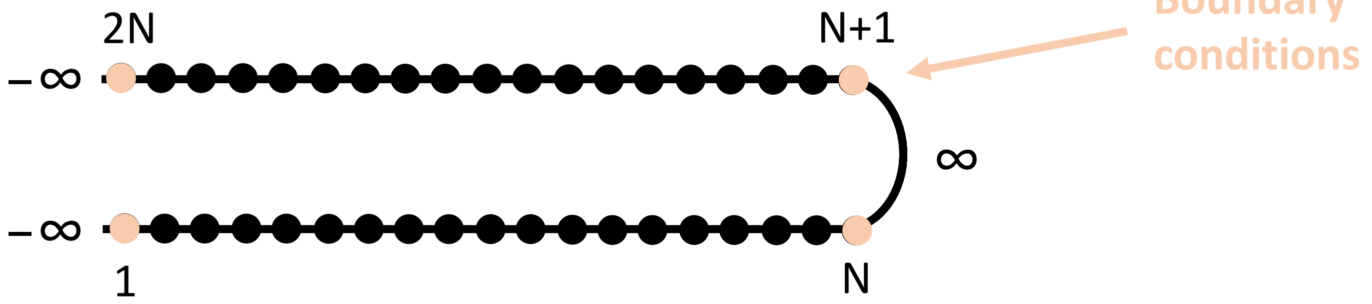



- Take expectation values at times on the Schwinger-Keldysh contour



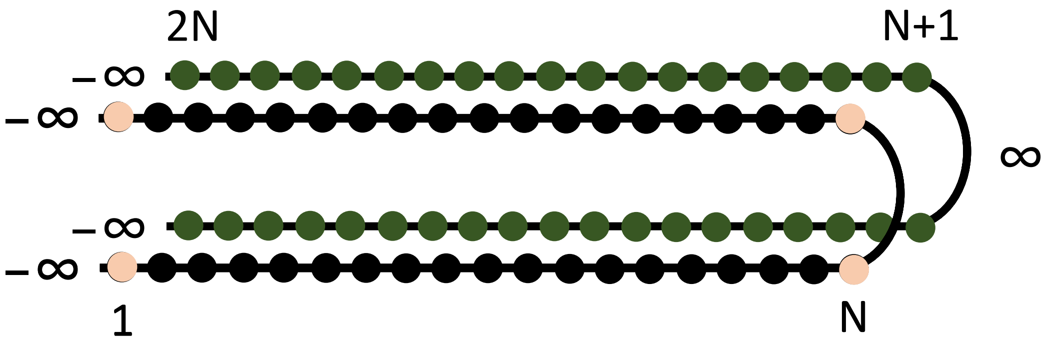

$$\rho(t) = U(t - t_i)\rho(t_i)U(t_i - t)$$

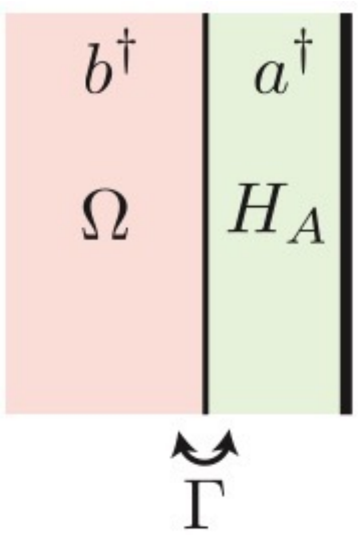
- Can express two-point correlators in terms of $G^R$, $G^A$ and $G^K$
- Discrete time contour



# Keldysh Action for Two Subsystems

- Total action $iS[\boldsymbol{a},\boldsymbol{b}] = iS_A[\boldsymbol{a}] + iS_B[\boldsymbol{b}] + iS_{AB}[\boldsymbol{a},\boldsymbol{b}]$ given by




$$iS_A[\boldsymbol{a}] = \int_\Omega d\Omega \left( -\bar{b}_{1,\Omega} b_{1,\Omega} + \sum_{i=2}^{2N} -\bar{b}_{i,\Omega} b_{i,\Omega} + \bar{b}_{i,\Omega} b_{i-1,\Omega} - i\Omega\delta t\, \bar{b}_{i,\Omega} b_{i-1,\Omega} \right)$$

$$iS_B[\boldsymbol{b}] = \int_\Omega d\Omega \left( -\bar{b}_{1,\Omega} b_{1,\Omega} + \rho_{\text{in}}(\Omega)\bar{b}_{1,\Omega} b_{2N,\Omega} + \sum_{i=2}^{2N} -\bar{b}_{i,\Omega} b_{i,\Omega} + \bar{b}_{i,\Omega} b_{i-1,\Omega} - i\Omega\delta t\, \bar{b}_{i,\Omega} b_{i-1,\Omega} \right)$$

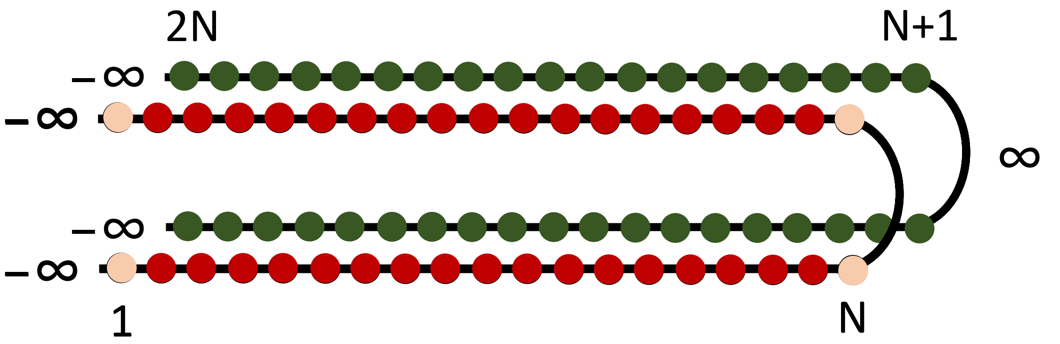
$$iS_{AB}[\boldsymbol{a},\boldsymbol{b}] = \int_\Omega d\Omega \sum_{i=2}^{2N} -i\delta t(\gamma_\Omega \bar{a}_i b_{i-1,\Omega} + \gamma^*_\Omega \bar{b}_{i,\Omega} a_{i-1})$$

Integrate out modes b
Take continuum limit

⇒ Continuum action with initial and final boundary conditions on the bath

# Daniel-Potts Continuum Action

- Continuum action from Aaron Daniel's thesis, University of Basel (2022)
- Discrete time contour for cavity (a) and free space (b) modes—integrate out all intermediate times on the b modes

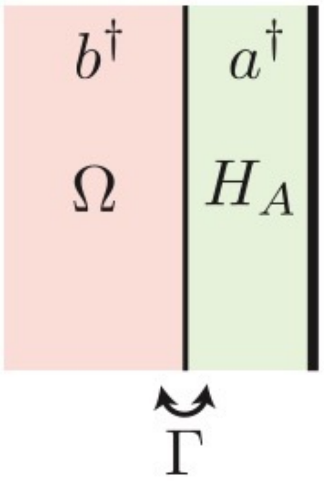



$$iS[a, b_{\text{in}}, b_{\text{out}}] = iS_A + iS_B + iS_{AB}$$

$$iS_A = i\int_{t_1}^{t_N} dt \left[\bar{a}^+(t)\left(i\partial_t + i\frac{\Gamma}{2}\right)a^+(t) - \bar{a}^-(t)\left(i\partial_t - i\frac{\Gamma}{2}\right)a^-(t) - H_A(\bar{a}^+, a^+) + H_A(\bar{a}^-, a^-)\right]$$

$$iS_B = -[\int_{t_1}^{t_N} dt \begin{pmatrix} \bar{b}_{\text{in}}^+(t) \\ \bar{b}_{\text{out}}^+(t) \\ \bar{b}_{\text{out}}^-(t) \\ \bar{b}_{\text{in}}^-(t) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_{\text{in}}^+(t) \\ b_{\text{out}}^+(t) \\ b_{\text{out}}^-(t) \\ b_{\text{in}}^-(t) \end{pmatrix}] + \ln(\rho_B(\bar{b}_{\text{in}}, b_{\text{in}}))$$

$$iS_{AB} = -\sqrt{\Gamma}\int_{t_1}^{t_N} dt\ \bar{a}^+(t)b_{\text{in}}^+(t) - \bar{b}_{\text{out}}^+(t)a^+(t) - \bar{a}^-(t)b_{\text{out}}^-(t) + \bar{b}_{\text{in}}^-(t)a^-(t)$$



# Output Field Generating Functional

- We want to generate moments of the output free-space modes, so we add a source field

$$iS_\psi = -i \int_{t_1}^{t_N} dt\ \bar{\psi}(t) b^{+}_{\text{out}}(t) + \bar{b}^{-}_{\text{out}}(t)\psi(t)$$

- Key difference from Lindblad: care about bath modes! We seed the bath with an initial state
- Now we integrate out everything except source fields to get the moment-generating function
  - Get Green's functions along the way
- Why Keldysh-Larkin-Ovchinnikov rotate?
  - Go from the contour-ordered Green's functions $G^T$,$G^{\tilde{T}}$,$G^<$,$G^>$ to the causal/anti-causal and Keldysh Green's functions $G^R$,$G^A$,$G^K$

$S[a, b_{\text{in}}, b_{\text{out}}]$

$\downarrow$ add source field $\psi b_{\text{out}}$

$S[a, b_{\text{in}}, b_{\text{out}}, \psi]$

$\downarrow$ integrate out $b_{\text{out}}$

$S[a, b_{\text{in}}, \psi]$

$\downarrow$ put in input state

$S[a, b_{\text{in}}, \psi]$

$\downarrow$ integrate out $b_{\text{in}}$

$S[a, \psi]$

$\downarrow$ Larkin rotate

$S[a, \psi] \to G^R,\ G^A,\ G^K$

$\downarrow$ integrate out $a$

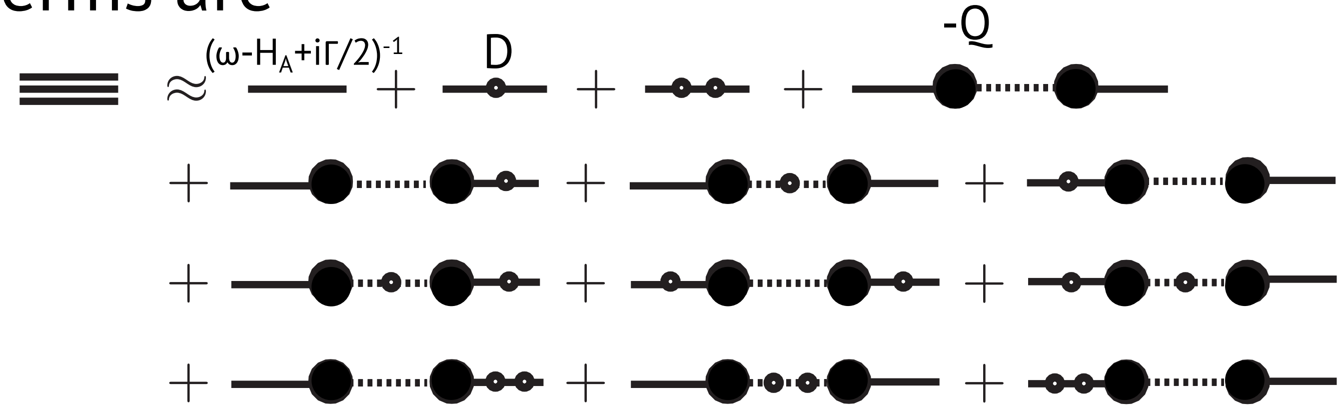
$S[\psi] \to$ calculate moments

# Diagrammatic Representation

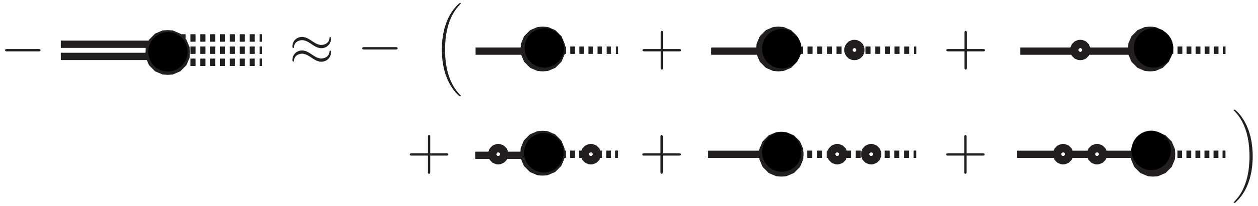
- Now, $(G^R)^{-1}$ is in the form so we can invert it perturbatively

$$G^R = 2\cdot\begin{pmatrix} \omega - H_A - \hat{D} + i\Gamma/2 & -\hat{Q} \\ -\hat{Q} & (\omega - H_A - \hat{D} + i\Gamma/2)^* \end{pmatrix}^{-1} = 2$$

where the terms are




and



# $g^{(1)}$ and $g^{(2)}$ Functionals



- $g^{(1)}$ functional $g^{(1)}(t,t') = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \rangle$

$$g^{(1)} = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t') \rangle = \int d\tau d\tau' \; (-i\Gamma G^R(t'-\tau') f(\tau') + f(\tau'))(i\Gamma G^A(t-\tau)\bar{f}(\tau) + \bar{f}(\tau))$$

- $g^{(2)}$ functional $g^{(2)}(t,t') = \dfrac{\langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle}{g^{(1)}(t)}$

$$\begin{aligned} g^{(1)} g^{(2)} &= \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle \\ &= \int d\tau d\tau' d\tau'' d\tau''' \; (-i\Gamma G^R(t'-\tau''') f(\tau''') + f(\tau'''))(i\Gamma G^A(t'-\tau'')\bar{f}(\tau'') + \bar{f}(\tau'')) \\ &\qquad \times (-i\Gamma G^R(t-\tau') f(\tau') + f(\tau'))(i\Gamma G^A(t-\tau)\bar{f}(\tau) + \bar{f}(\tau)) \end{aligned}$$

- Completing the time integrals gives δ(t-t’) which creates loop diagrams

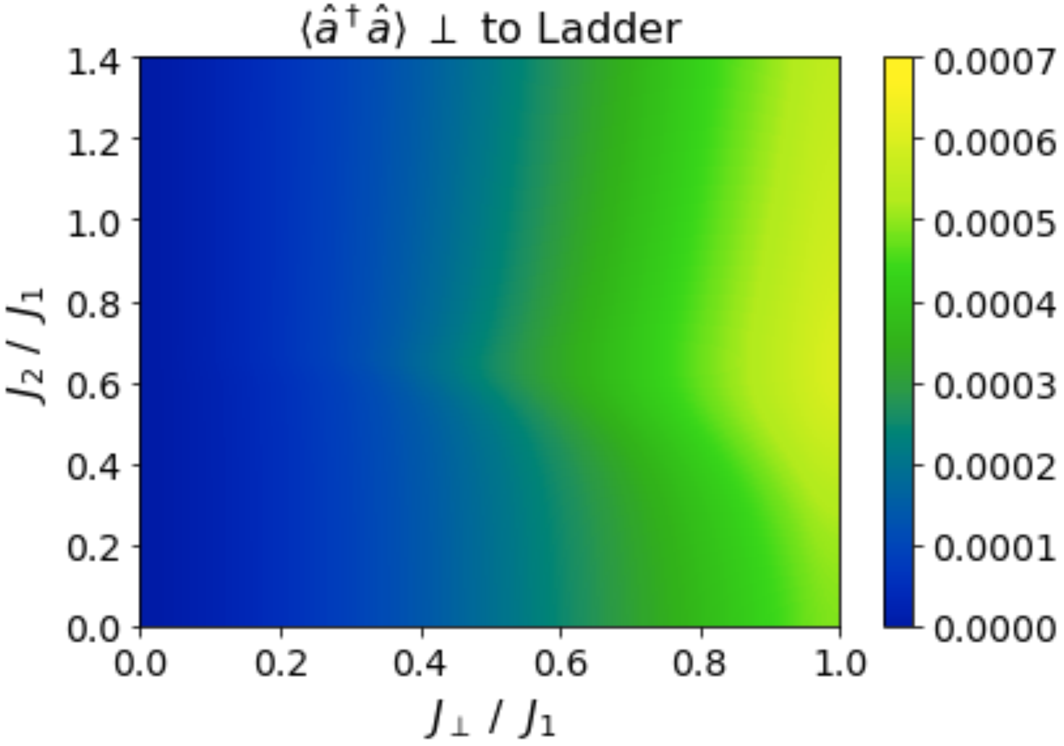
# Example: Output Photons and Cavity Correlations

DMRG work by Ben Kass

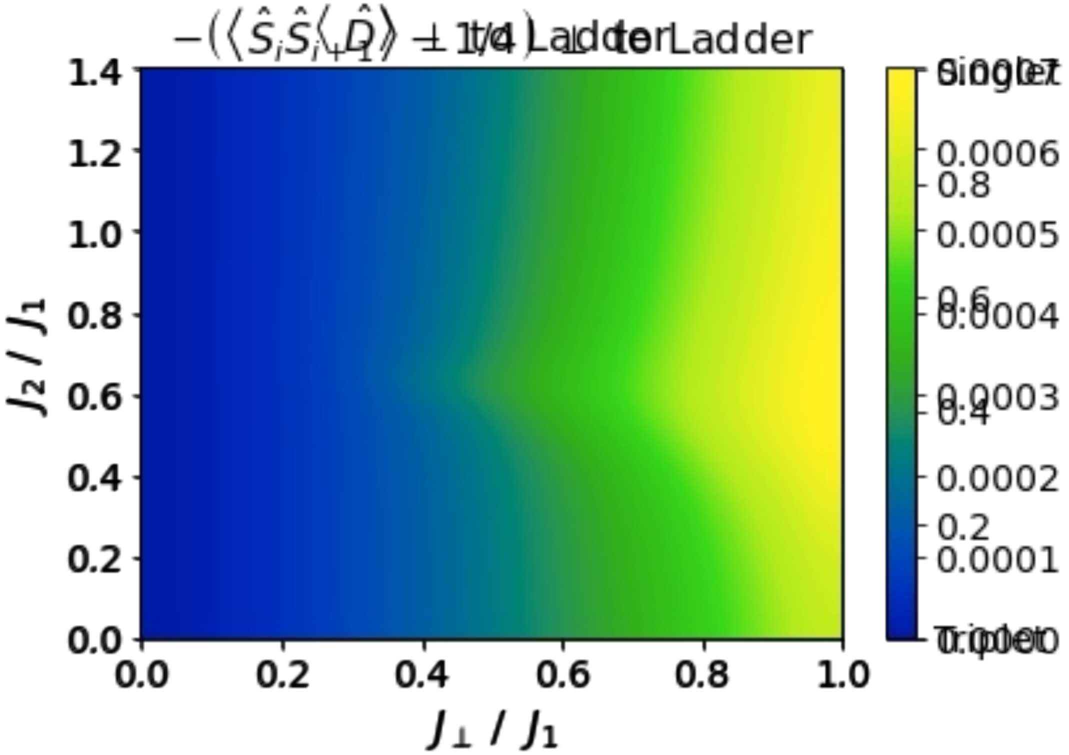
- Input Fock state with D coupling to the material
- $J_1$-$J_2$ spin ladder in a cavity
  - Finite size DMRG on a 6x2 spin ladder

$$\hat{D} = \sum_{ij} C_{ij,\mu} \left( \hat{S}_i \cdot \hat{S}_j - \frac{1}{4} \right)$$

Photon Number:



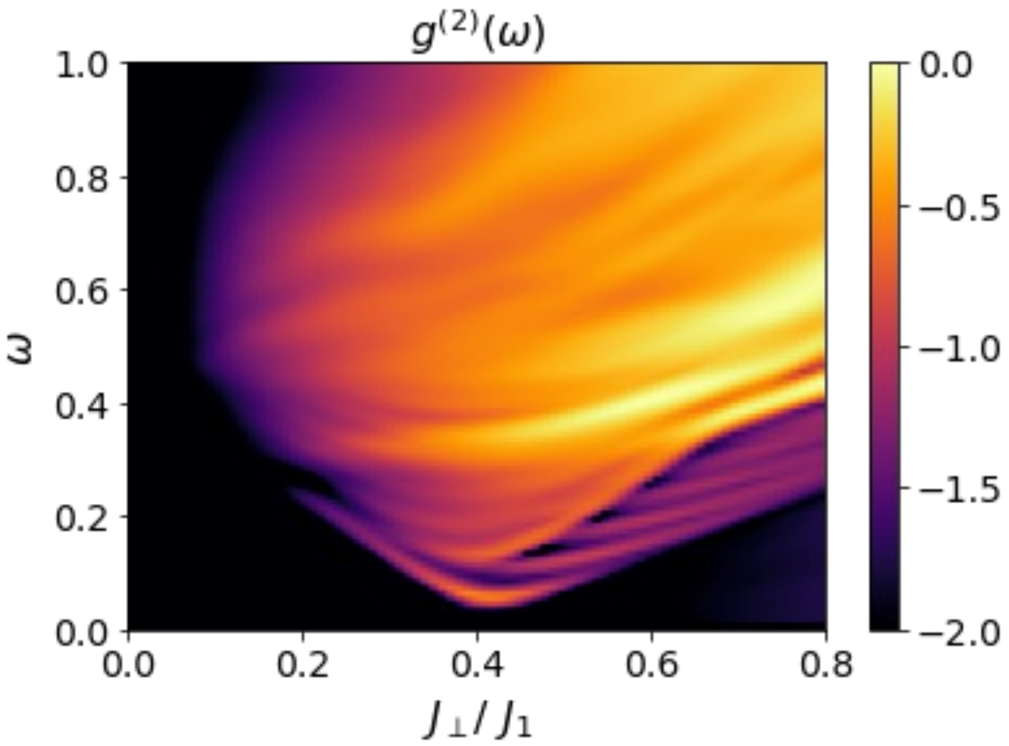

Predicted Photon Number:

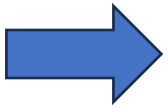


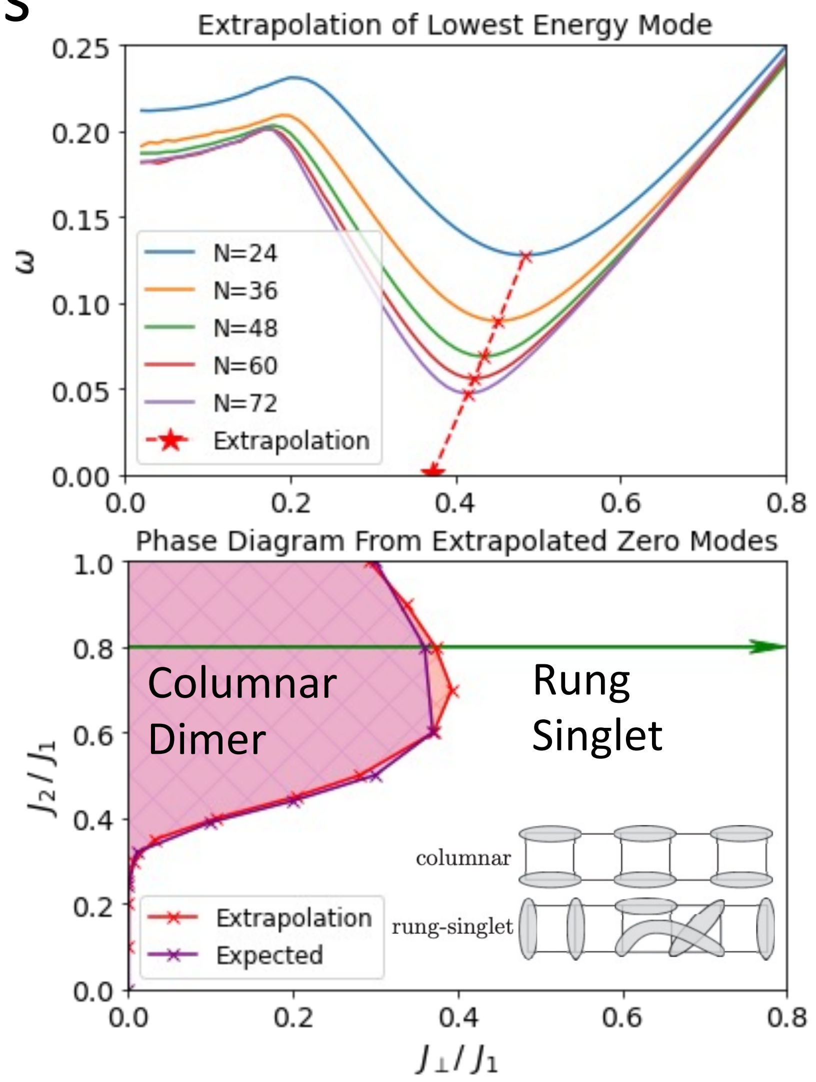
# Extrapolating Phase Diagram from Output Light

DMRG work by Ben Kass

- Obtain phase diagram from photon fluctuations
- DMRG with Chebyshev polynomial expansion
  - 72x2 ladder for $J_2/J_1 = 0.8$

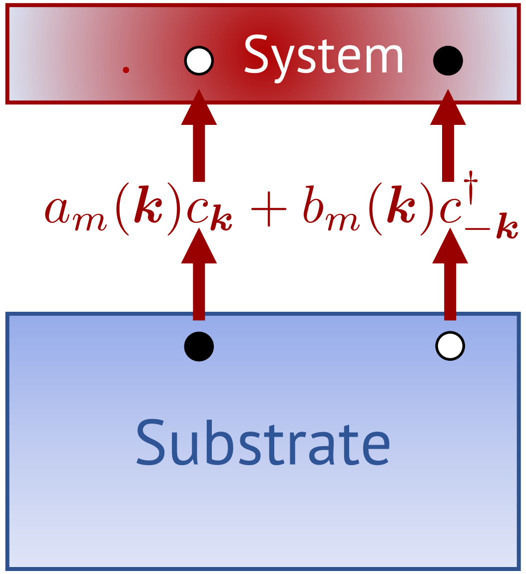


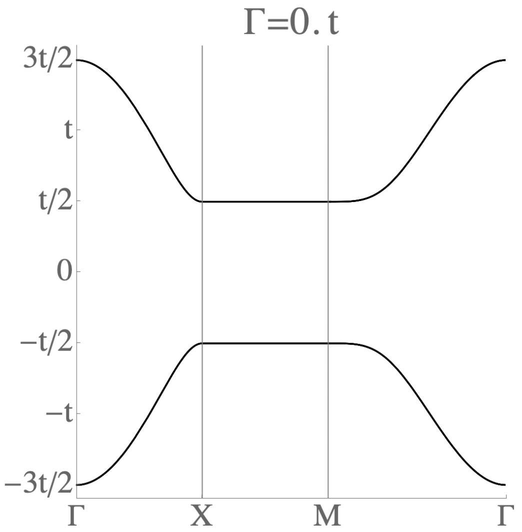


## Band Structure Engineering

- Physical Review B 106, 161109
- Dissipation-free dark space
- Long lived flat bands

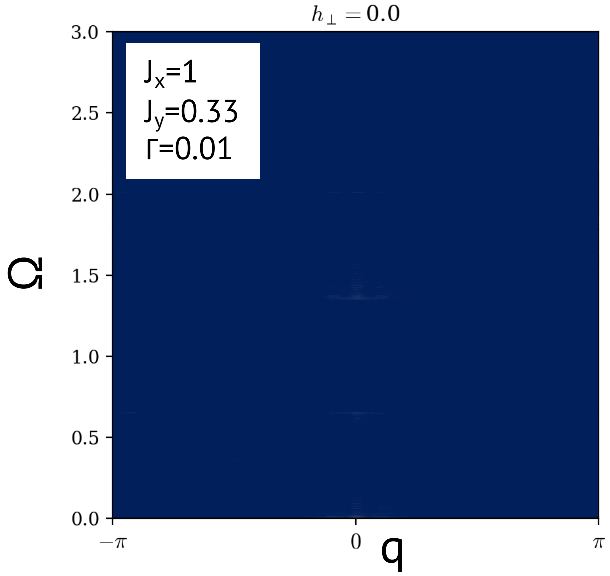



Flat Bands in QWZ Model

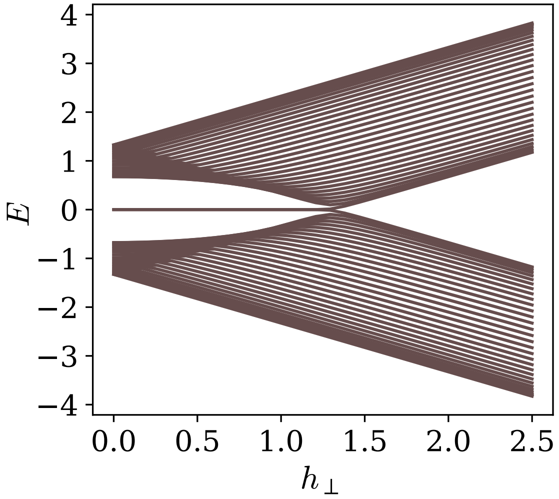



## Dynamic Response of Open Systems

- arXiv:2401.17368, 2402.06593
- Lindblad-Keldysh Green functions
- Finite frequency probes of $\rho_{ss}$
- Bose, fermi, and spin systems

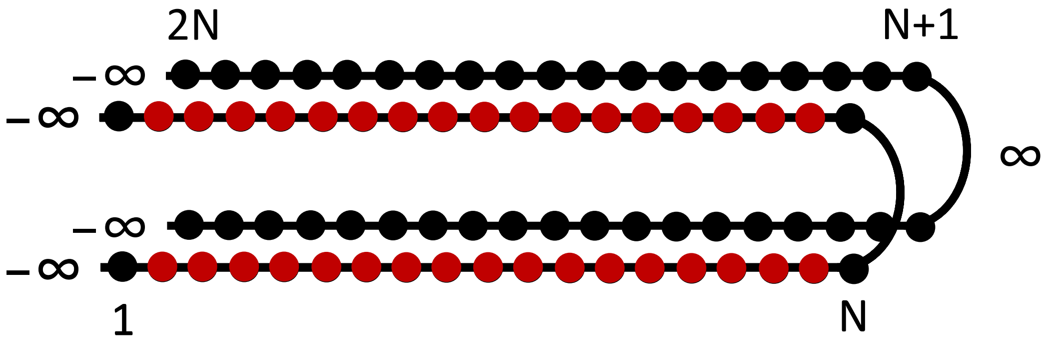


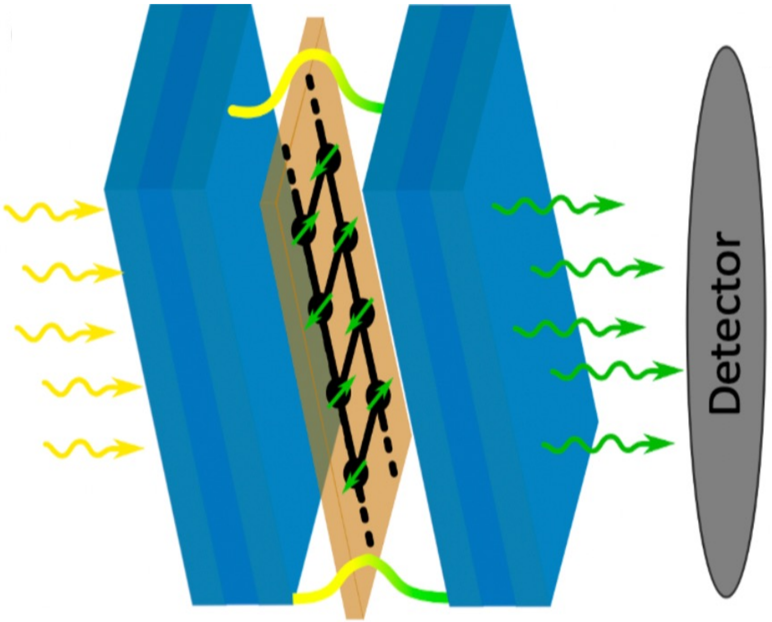

Dissipative XY Model



# Outlook

- Cavity quantum materials are emerging as a promising platform for exotic physics
- Quantum photon spectroscopy is natural for probing this physics and gives different information than classical light spectroscopy
- Here: Keldysh input-output theory is a natural language to calculate output correlation functions from leaky cavities
- Output photon fields can be used to reconstruct orders in the cavity (Ben Kass)




- Preprint coming soon!

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# …

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# Coherent Input State

- After completing these steps for an input zero-temperature coherent state $\rho_B = \bigotimes_\Omega e^{-|\alpha_\Omega|^2} |\alpha_\Omega\rangle\langle\alpha_\Omega|$ with $f(t) = \int_\Omega e^{-i\Omega(t-t_1)} \alpha_\Omega$ we find for $H_A = \Omega_A$

$$iS^{\text{mat}} = i\int_{t_1}^{t_N} dt \begin{pmatrix} \bar{a}^{\text{cl}} \\ \bar{a}^{\text{q}} \end{pmatrix} \begin{pmatrix} 0 & i\partial_t - \Omega_A - i\Gamma/2 \\ i\partial_t - \Omega_A + i\Gamma/2 & i\Gamma \end{pmatrix} \begin{pmatrix} a^{\text{cl}} \\ a^{\text{q}} \end{pmatrix}$$

$$iS^{\text{vec}} = i\int_{t_1}^{t_N} dt \begin{pmatrix} \bar{a}^{\text{cl}} \\ \bar{a}^{\text{q}} \end{pmatrix} \cdot \begin{pmatrix} -\sqrt{\Gamma}\psi\frac{1}{\sqrt{2}} \\ i\sqrt{\Gamma}\sqrt{2}f - \sqrt{\Gamma}\psi\frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} -\sqrt{\Gamma}\bar{\psi}\frac{1}{\sqrt{2}} \\ -i\sqrt{\Gamma}\sqrt{2}\bar{f} - \sqrt{\Gamma}\bar{\psi}\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a^{\text{cl}} \\ a^{\text{q}} \end{pmatrix}$$

$$iS^{\text{extra}} = i\int_{t_1}^{t_N} dt \ -\bar{\psi}f - \bar{f}\psi$$

- Now let us couple the cavity modes to the material

$$H_A = \Omega_A + \hat{D}a^\dagger a + \hat{Q}(aa + a^\dagger a^\dagger)$$

density coupling *D* and pair gain/loss *Q*

# A Diversion About Matrices & Green Functions



- For n x n blocks A and B the matrix inverse is (for invertible A)

$$\begin{pmatrix} A & B \\ B & A^* \end{pmatrix}^{-1} = \begin{pmatrix} (A - B(A^*)^{-1}B)^{-1} & -A^{-1}B(A^* - BA^{-1}B)^{-1} \\ -(A^*)^{-1}B(A - B(A^*)^{-1}B)^{-1} & (A^* - BA^{-1}B)^{-1} \end{pmatrix}$$

where we can expand for small $A^{-1}B$ (geometric series expansion)

$$\frac{1}{A - B(A^*)^{-1}B} = \frac{1}{1 - A^{-1}B(A^*)^{-1}B}\frac{1}{A} = (1+A^{-1}B(A^*)^{-1}B+(A^{-1}B(A^*)^{-1}B)^2+\dots)A^{-1}$$

and for our purposes $A^{-1}$ can be expanded as well

$$A^{-1} = \frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A - \hat{D}} = \frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A} + \frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A}\hat{D}\frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A} + \frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A}\hat{D}\frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A}\hat{D}\frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A} + \dots$$

- We can express this diagrammatically

$$G_A = \frac{1}{\omega\hat{1} - \hat{\mathcal{H}}_A}$$

$$A^{-1} = G_A + G_A D G_A + G_A D G_A D G_A + \dots$$