

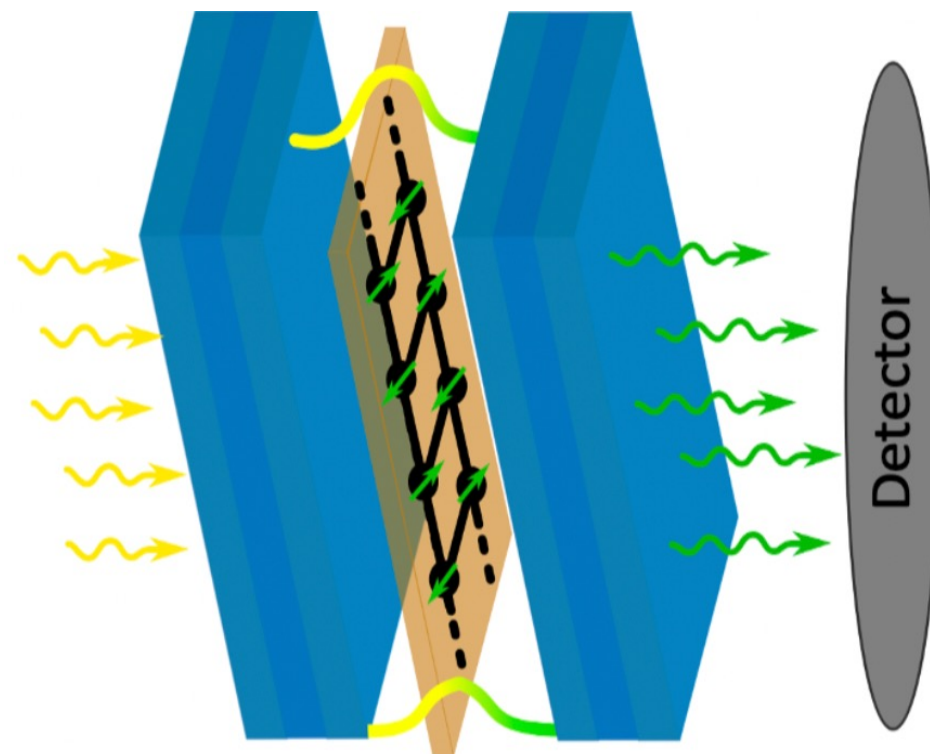
Keldysh Input-Output Theory: Application to Cavity Quantum Materials

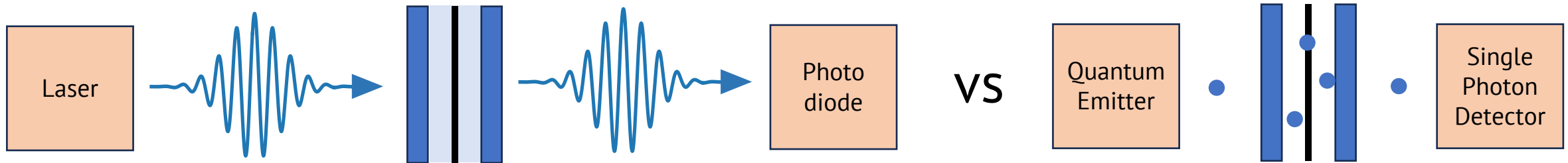
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with Ben Kass and Martin Claassen (also U. Pennsylvania)

18 April 2024 • 2024 Capri Spring School:

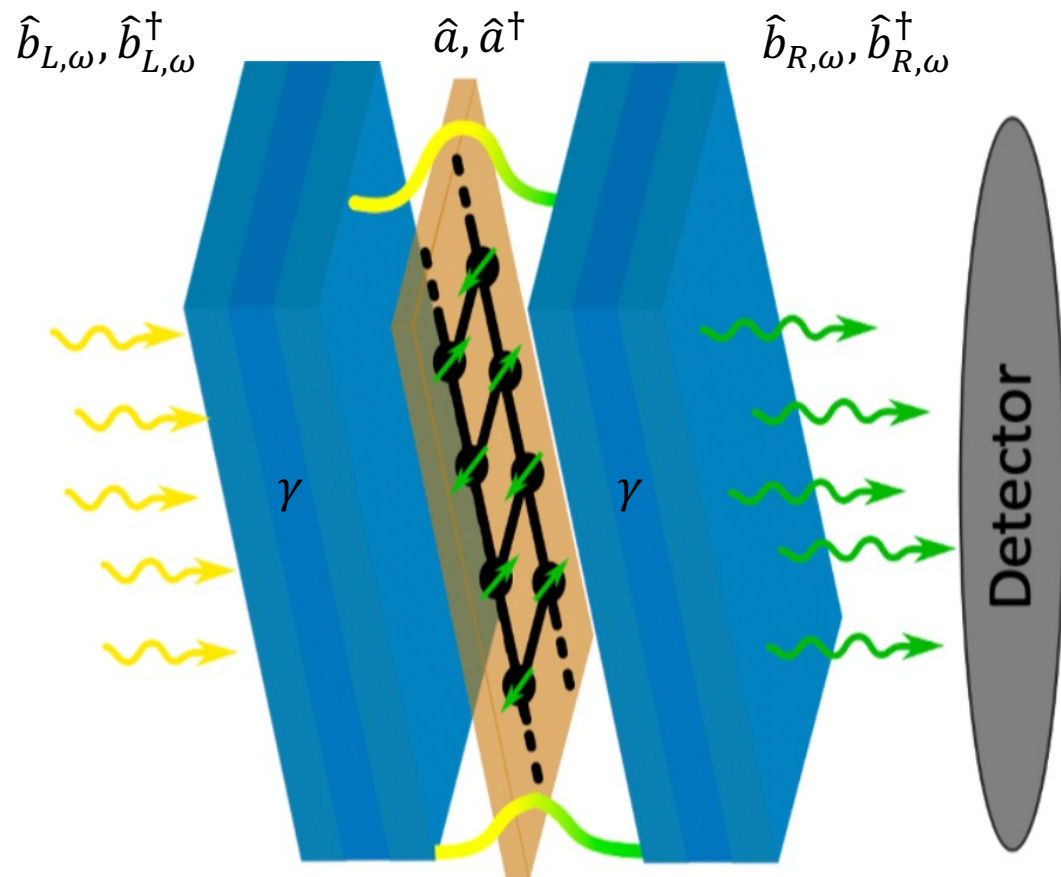
Non-unitary many-body quantum dynamics



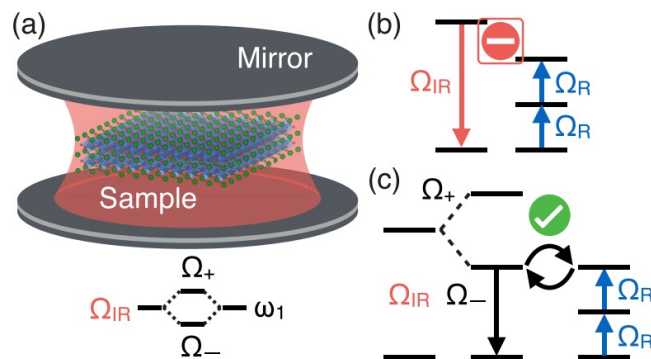


- Quantum light spectroscopy is a different probe than classical light
 - Few photon input and output states, possibly with entanglement
 - Don't destroy the state and can probe subtle quantum superpositions of states
 - In a cavity we can have multiple scattering processes
 - Note: neutron and x-ray scattering just have one scattering between source and detector
- Additionally we can convert from the correlation functions of the material in the cavity to the experimentally observable output fields
- Here: Keldysh input-output theory as a way to calculate input-output correlation functions for cavity quantum materials with quantum light

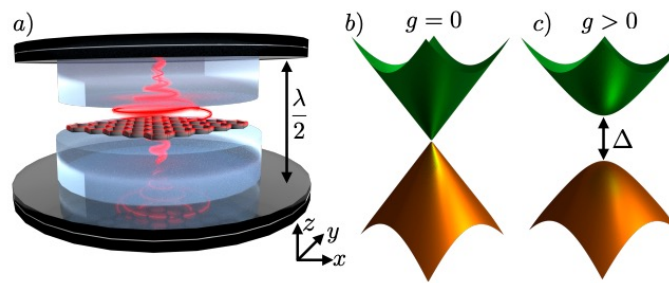
- Many recent proposals on cavity quantum materials for exotic physics
- How to probe states?



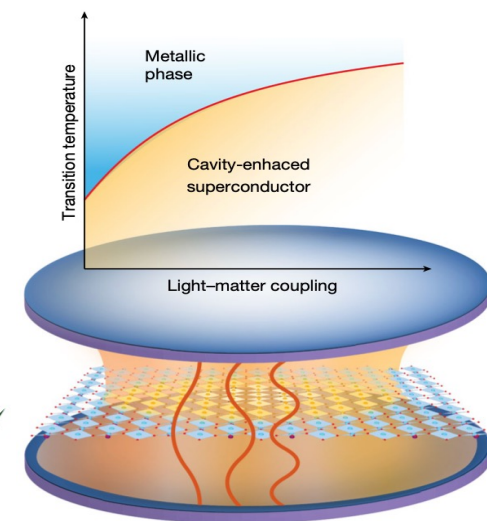
Nonlinear phononics
PRR 3, L032046 (2019)



Cavity induced Chern insulators
PRB 99, 235156 (2019)

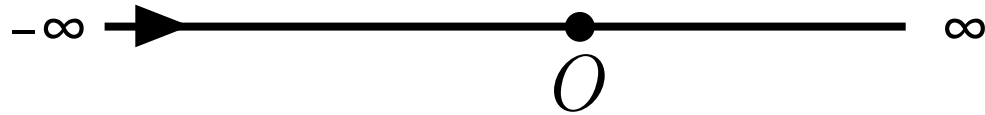


Metal-superconductor transition
Nature 606, 41 (2022)

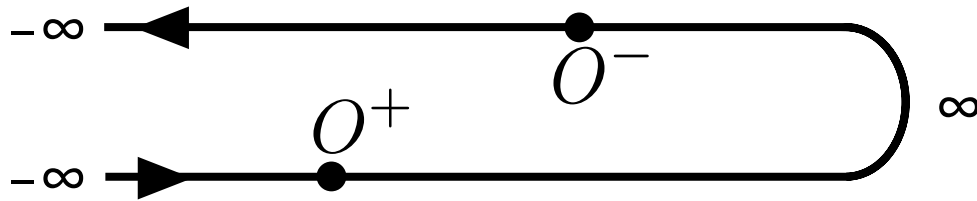


- Standard quantum optical input-output relations
 - Gardiner and Collett PRA 31, 3761 (1985)
 - Solve equations of motion for bath field operators: empty cavity with leakiness γ
$$\hat{b}_{\text{out}}(t) = \hat{b}_{\text{in}}(t) + \gamma \hat{a}(t)$$
- Generalize to extended systems in a cavity (vs few quantum emitters)
- What is missing that we gain with field theory?
 - Rigorous perturbation theory in powers of many-body operators
 - E.g. $g^{(1)}$ and $g^{(2)}$ functionals in terms of pair gain/loss operators coupling the material to cavity modes
 - Effective descriptions of collective modes
 - E.g. superconducting modes from fermions
 - E.g. magnons from spins

- Familiar equilibrium contour

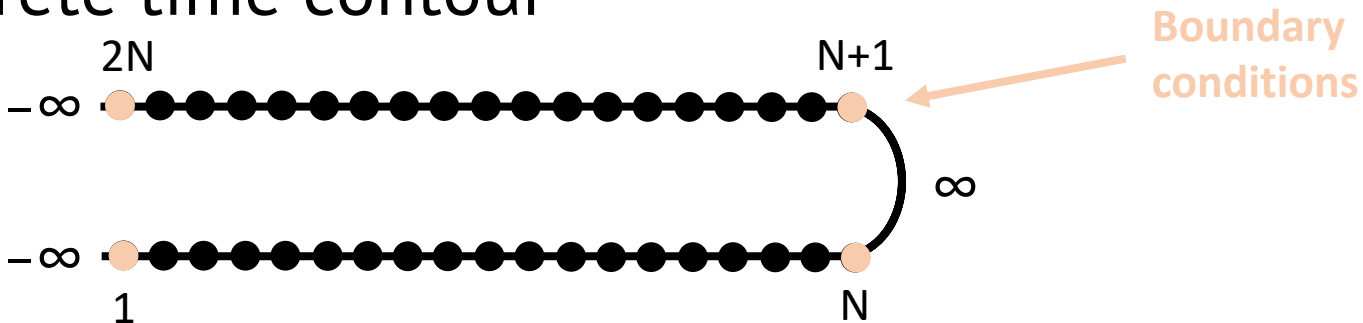


- Take expectation values at times on the Schwinger-Keldysh contour



$$\rho(t) = U(t - t_i)\rho(t_i)U(t_i - t)$$

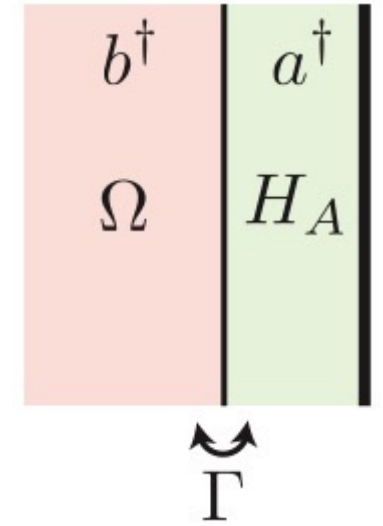
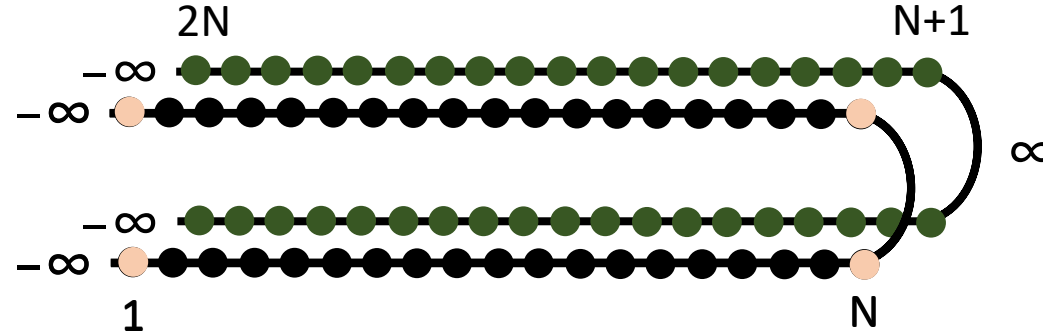
- Can express two-point correlators in terms of G^R , G^A and G^K
- Discrete time contour



Keldysh Action for Two Subsystems

1

- Total action $iS[\mathbf{a}, \mathbf{b}] = iS_A[\mathbf{a}] + iS_B[\mathbf{b}] + iS_{AB}[\mathbf{a}, \mathbf{b}]$ given by



$$iS_A[\mathbf{a}] = \int_{\Omega} d\Omega \left(-\bar{b}_{1,\Omega} b_{1,\Omega} + \sum_{i=2}^{2N} -\bar{b}_{i,\Omega} b_{i,\Omega} + \bar{b}_{i,\Omega} b_{i-1,\Omega} - i\Omega \delta t \bar{b}_{i,\Omega} b_{i-1,\Omega} \right)$$

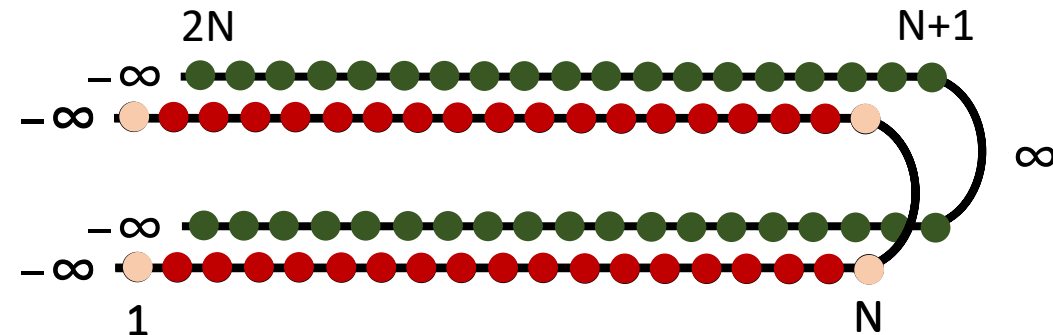
$$iS_B[\mathbf{b}] = \int_{\Omega} d\Omega \left(-\bar{b}_{1,\Omega} b_{1,\Omega} + \rho_{\text{in}}(\Omega) \bar{b}_{1,\Omega} b_{2N,\Omega} + \sum_{i=2}^{2N} -\bar{b}_{i,\Omega} b_{i,\Omega} + \bar{b}_{i,\Omega} b_{i-1,\Omega} - i\Omega \delta t \bar{b}_{i,\Omega} b_{i-1,\Omega} \right)$$

$$iS_{AB}[\mathbf{a}, \mathbf{b}] = \int_{\Omega} d\Omega \sum_{i=2}^{2N} -i\delta t (\gamma_{\Omega} \bar{a}_i b_{i-1,\Omega} + \gamma_{\Omega}^* \bar{b}_{i,\Omega} a_{i-1})$$

Integrate out modes b
Take continuum limit

Continuum action with initial and final boundary conditions on the bath

- Continuum action from Aaron Daniel's thesis, University of Basel (2022)
- Discrete time contour for cavity (a) and free space (b) modes—integrate out all intermediate times on the b modes

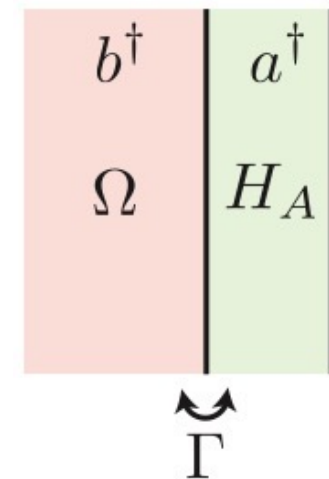


$$iS_{\text{out}}[a, b_{\text{in}}, b_{\text{out}}, \psi] = iS_A + iS_B + iS_{AB} + iS_\psi$$

$$iS_A = i \int_{t_1}^{t_N} dt \left[\bar{a}^+(t) \left(i\partial_t + i\frac{\Gamma}{2} \right) a^+(t) - \bar{a}^-(t) \left(i\partial_t - i\frac{\Gamma}{2} \right) a^-(t) - H_A(\bar{a}^+, a^+) + H_A(\bar{a}^-, a^-) \right]$$

$$iS_B = - \left[\int_{t_1}^{t_N} dt \begin{pmatrix} \bar{b}_{\text{in}}^+(t) \\ \bar{b}_{\text{out}}^+(t) \\ \bar{b}_{\text{out}}^-(t) \\ \bar{b}_{\text{in}}^-(t) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_{\text{in}}^+(t) \\ b_{\text{out}}^+(t) \\ b_{\text{out}}^-(t) \\ b_{\text{in}}^-(t) \end{pmatrix} \right] + \ln(\rho_B(\bar{b}_{\text{in}}, b_{\text{in}}))$$

$$iS_{AB} = -\sqrt{\Gamma} \int_{t_1}^{t_N} dt \left[\bar{a}^+(t) b_{\text{in}}^+(t) - \bar{b}_{\text{out}}^+(t) a^+(t) - \bar{a}^-(t) b_{\text{out}}^-(t) + \bar{b}_{\text{in}}^-(t) a^-(t) \right]$$



- We want to generate moments of the output free-space modes, so we add a source field

$$iS_\psi = -i \int_{t_1}^{t_N} dt \bar{\psi}(t) b_{\text{out}}^+(t) + \bar{b}_{\text{out}}^-(t) \psi(t)$$

- Key difference from Lindblad: care about bath modes! We seed the bath with an initial state
- Now we integrate out everything except source fields to get the moment-generating function
 - Get Green's functions along the way
- Why Keldysh-Larkin-Ovchinnikov rotate?
 - Go from the contour-ordered Green's functions $G^T, G^{\tilde{T}}, G^<, G^>$ to the causal/anti-causal and Keldysh Green's functions G^R, G^A, G^K

$$S[a, b_{\text{in}}, b_{\text{out}}]$$

↓ add source field ψb_{out}

$$S[a, b_{\text{in}}, b_{\text{out}}, \psi]$$

↓ integrate out b_{out}

$$S[a, b_{\text{in}}, \psi]$$

↓ put in input state

$$S[a, b_{\text{in}}, \psi]$$

↓ integrate out b_{in}

$$S[a, \psi]$$

↓ Larkin rotate

$$S[a, \psi] \rightarrow G^R, G^A, G^K$$

↓ integrate out a

$$S[\psi] \rightarrow \text{calculate moments}$$

- Now, $(G^R)^{-1}$ is in the form so we can invert it perturbatively

$$G^R = 2 \cdot \begin{pmatrix} \omega - H_A - \hat{D} + i\Gamma/2 & -\hat{Q} \\ -\hat{Q} & (\omega - H_A - \hat{D} + i\Gamma/2)^* \end{pmatrix}^{-1} = 2 \left(\begin{array}{c} \equiv \\ - \equiv \bullet \text{dotted} \end{array} \right)$$

where the terms are

$$\begin{aligned} \equiv & \approx \frac{(\omega - H_A + i\Gamma/2)^{-1}}{\text{---}} + \text{---} \overset{D}{\bullet} \text{---} + \text{---} \bullet \bullet \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \\ & + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} \\ & + \text{---} \bullet \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} \\ & + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} \end{aligned}$$

and

$$\begin{aligned} - \equiv \bullet \text{dotted} & \approx - \left(\text{---} \bullet \text{---} \text{dotted} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} \right. \\ & \left. + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{dotted} \bullet \text{---} \text{---} \right) \end{aligned}$$

- $g^{(1)}$ functional $g^{(1)}(t, t') = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t') \rangle$

$$g^{(1)} = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t') \rangle = \int d\tau d\tau' (-i\Gamma G^R(t' - \tau') f(\tau') + f(\tau')) (i\Gamma G^A(t - \tau) \bar{f}(\tau) + \bar{f}(\tau))$$

- $g^{(2)}$ functional $g^{(2)}(t, t') = \frac{\langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle}{g^{(1)}(t)}$

$$\begin{aligned} g^{(1)} g^{(2)} &= \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle \\ &= \int d\tau d\tau' d\tau'' d\tau''' (-i\Gamma G^R(t' - \tau''') f(\tau''') + f(\tau''')) (i\Gamma G^A(t' - \tau'') \bar{f}(\tau'') + \bar{f}(\tau'')) \\ &\quad \times (-i\Gamma G^R(t - \tau') f(\tau') + f(\tau')) (i\Gamma G^A(t - \tau) \bar{f}(\tau) + \bar{f}(\tau)) \end{aligned}$$

- Completing the time integrals gives $\delta(t-t')$ which creates loop diagrams

Example: Output Photons and Cavity Correlations

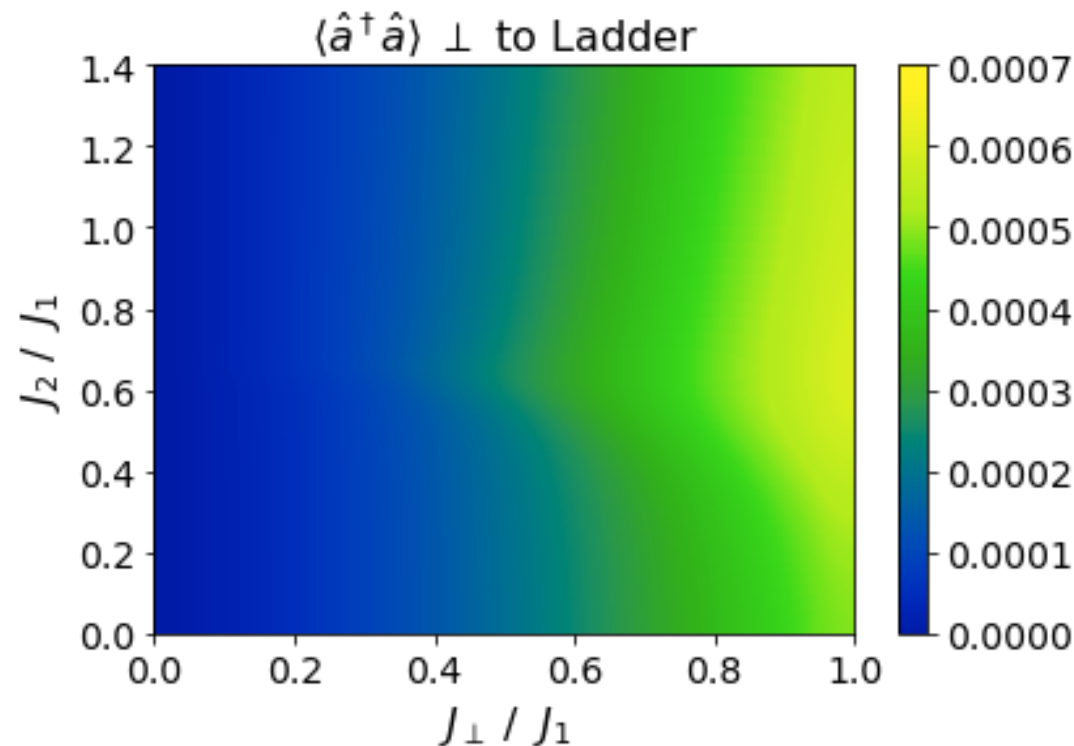
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DMRG work by Ben Kass

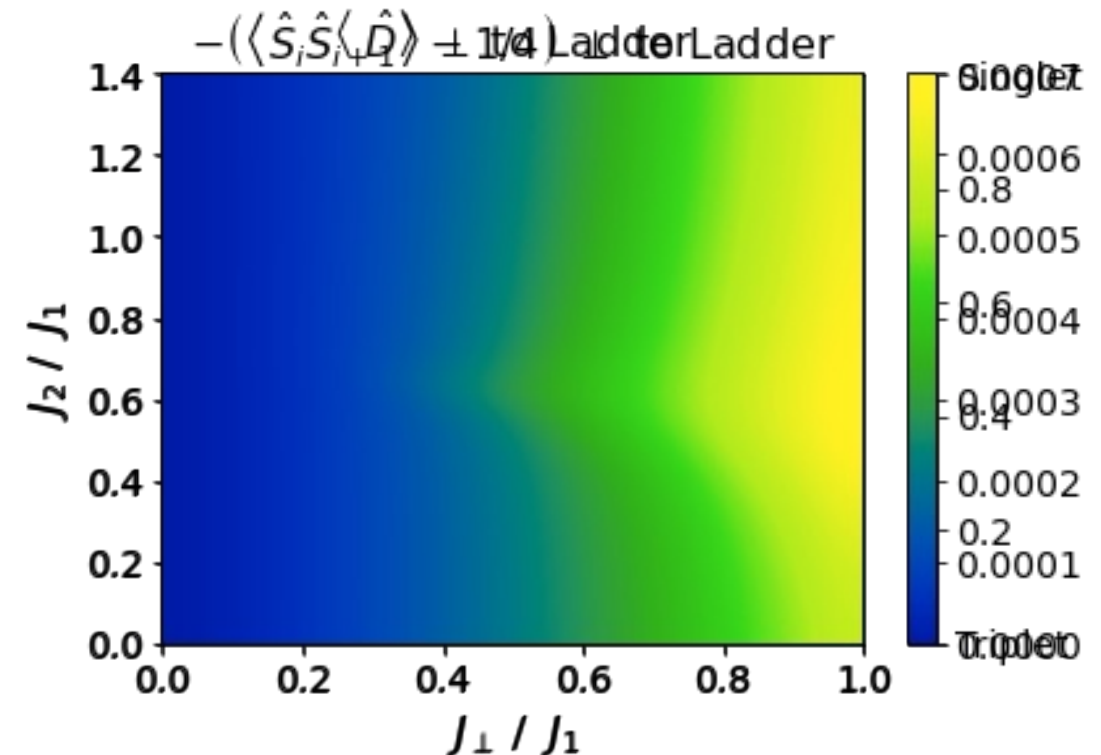
- Input Fock state with D coupling to the material
- J_1 - J_2 spin ladder in a cavity
 - Finite size DMRG on a 6x2 spin ladder

$$\hat{D} = \sum_{ij} C_{ij,\mu} \left(\hat{S}_i \cdot \hat{S}_j - \frac{1}{4} \right)$$

Photon Number:



Spin-Difference Photon Number:

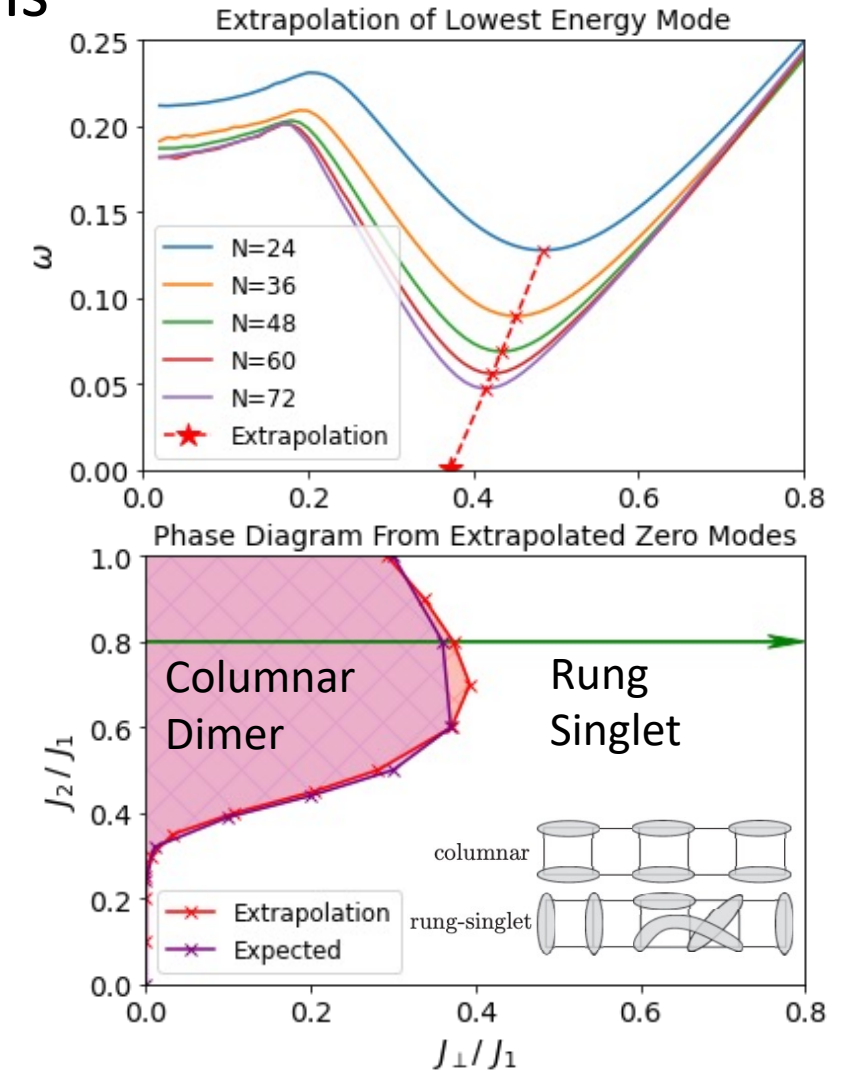
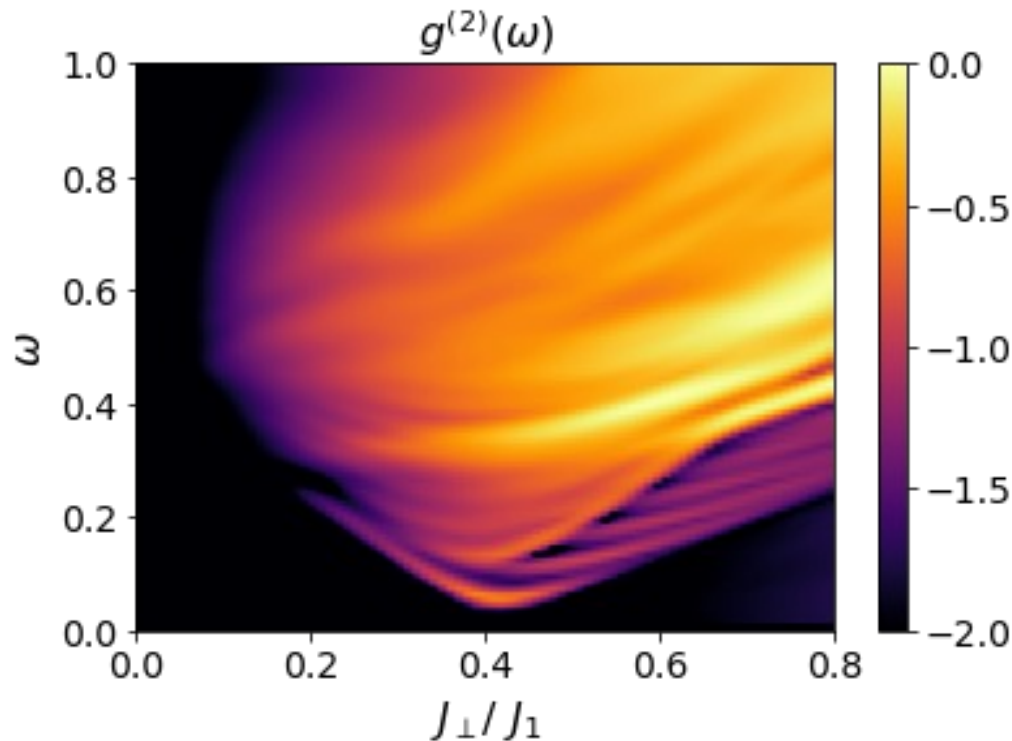


Extrapolating Phase Diagram from Output Light

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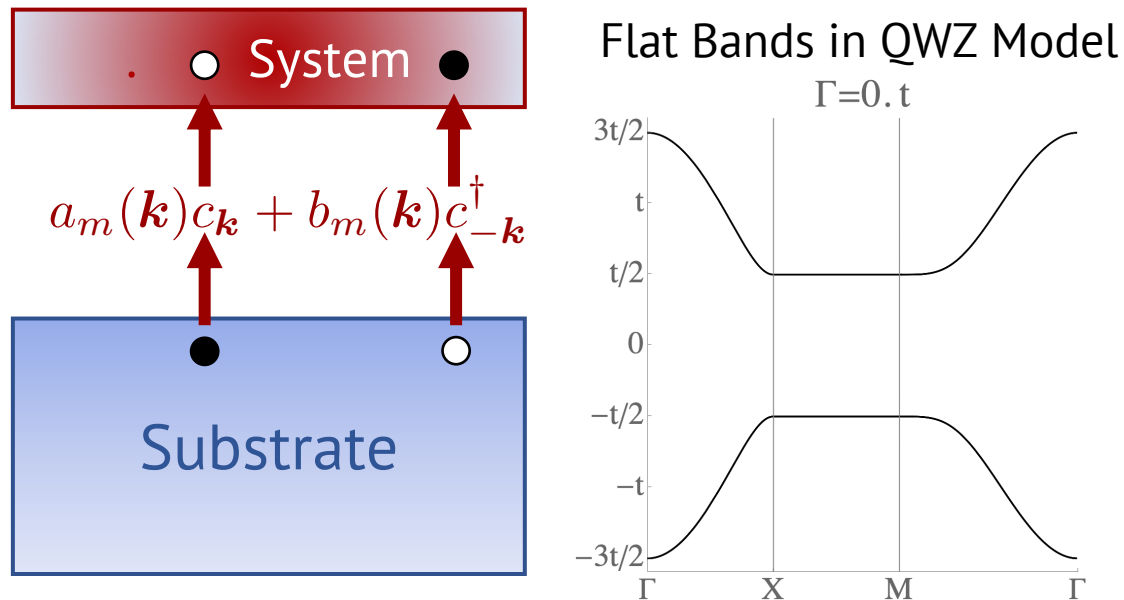
- Obtain phase diagram from photon fluctuations
- DMRG with Chebyshev polynomial expansion
 - 72x2 ladder for $J_2/J_1 = 0.8$

DMRG work by Ben Kass



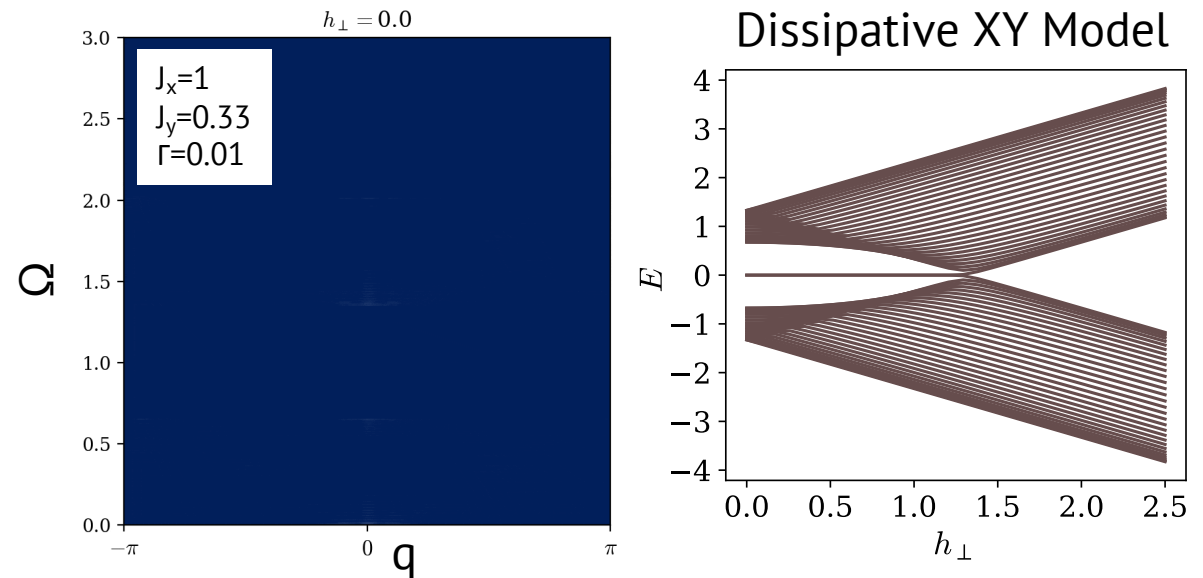
Band Structure Engineering

- Physical Review B 106, 161109
- Dissipation-free dark space
- Long lived flat bands

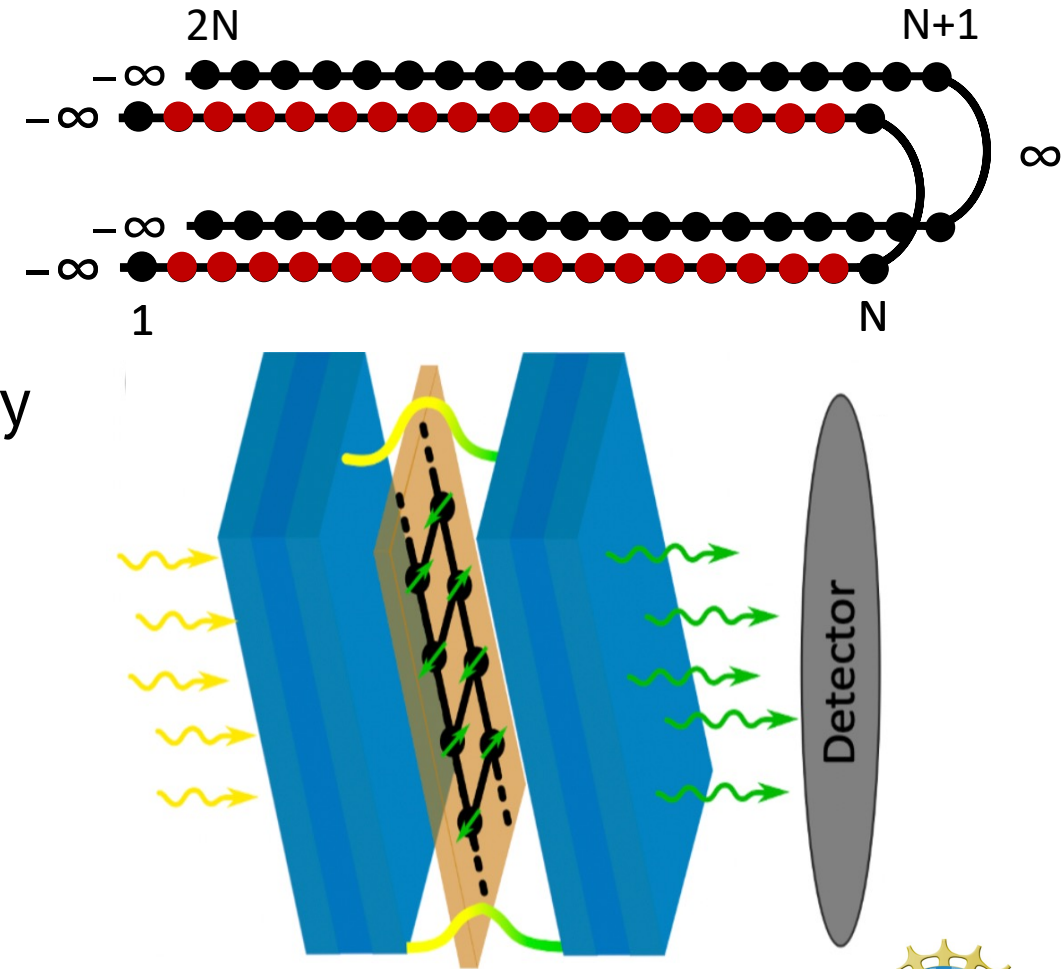


Dynamic Response of Open Systems

- arXiv:2401.17368, 2402.06593
- Lindblad-Keldysh Green functions
- Finite frequency probes of ρ_{ss}
- Bose, fermi, and spin systems



- Cavity quantum materials are emerging as a promising platform for exotic physics
- Quantum photon spectroscopy is natural for probing this physics and gives different information than classical light spectroscopy
- Here: Keldysh input-output theory is a natural language to calculate output correlation functions from leaky cavities
- Output photon fields can be used to reconstruct orders in the cavity (Ben Kass)
- Preprint coming soon!



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...

● ...

- After completing these steps for an input zero-temperature coherent state $\rho_B = \bigotimes_{\Omega} e^{-|\alpha_{\Omega}|^2} |\alpha_{\Omega}\rangle \langle \alpha_{\Omega}|$ with $f(t) = \int_{\Omega} e^{-i\Omega(t-t_1)} \alpha_{\Omega}$ we find for $H_A = \Omega_A$

$$iS^{\text{mat}} = i \int_{t_1}^{t_N} dt \begin{pmatrix} \bar{a}^{\text{cl}} \\ \bar{a}^{\text{q}} \end{pmatrix} \begin{pmatrix} 0 & i\partial_t - \Omega_A - i\Gamma/2 \\ i\partial_t - \Omega_A + i\Gamma/2 & i\Gamma \end{pmatrix} \begin{pmatrix} a^{\text{cl}} \\ a^{\text{q}} \end{pmatrix}$$

$$iS^{\text{vec}} = i \int_{t_1}^{t_N} dt \begin{pmatrix} \bar{a}^{\text{cl}} \\ \bar{a}^{\text{q}} \end{pmatrix} \cdot \begin{pmatrix} -\sqrt{\Gamma}\psi \frac{1}{\sqrt{2}} \\ i\sqrt{\Gamma}\sqrt{2}f - \sqrt{\Gamma}\psi \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} -\sqrt{\Gamma}\bar{\psi} \frac{1}{\sqrt{2}} \\ -i\sqrt{\Gamma}\sqrt{2}\bar{f} - \sqrt{\Gamma}\bar{\psi} \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a^{\text{cl}} \\ a^{\text{q}} \end{pmatrix}$$

$$iS^{\text{extra}} = i \int_{t_1}^{t_N} dt -\bar{\psi}f - \bar{f}\psi$$

- Now let us couple the cavity modes to the material

$$H_A = \Omega_A + \hat{D}a^{\dagger}a + \hat{Q}(aa + a^{\dagger}a^{\dagger})$$

density coupling D and pair gain/loss Q

- For $n \times n$ blocks A and B the matrix inverse is (for invertible A)

$$\begin{pmatrix} A & B \\ B & A^* \end{pmatrix}^{-1} = \begin{pmatrix} (A - B(A^*)^{-1}B)^{-1} & -A^{-1}B(A^* - BA^{-1}B)^{-1} \\ -(A^*)^{-1}B(A - B(A^*)^{-1}B)^{-1} & (A^* - BA^{-1}B)^{-1} \end{pmatrix}$$

where we can expand for small $A^{-1}B$ (geometric series expansion)

$$\frac{1}{A - B(A^*)^{-1}B} = \frac{1}{1 - A^{-1}B(A^*)^{-1}B} \frac{1}{A} = (1 + A^{-1}B(A^*)^{-1}B + (A^{-1}B(A^*)^{-1}B)^2 + \dots) A^{-1}$$

and for our purposes A^{-1} can be expanded as well

$$A^{-1} = \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A - \hat{D}} = \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} + \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} \hat{D} \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} + \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} \hat{D} \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} \hat{D} \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A} + \dots$$

- We can express this diagrammatically

$$G_A = \frac{1}{\omega \hat{1} - \hat{\mathcal{H}}_A}$$

$$\begin{aligned} A^{-1} &= G_A + G_A D G_A + G_A D G_A D G_A + \dots \\ &= \text{---} + \text{---} \overset{D}{\bullet} \text{---} + \text{---} \overset{D}{\bullet} \overset{D}{\bullet} \text{---} + \dots \end{aligned}$$