Derivation of the wave equation for light

Recall Maxwell's Equations:

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1}$$

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{2}$$

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{3}$$

$$\nabla \cdot B = 0 \tag{4}$$

In free space there are no charge densities or currents, so:

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{5}$$

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{6}$$

$$\nabla \cdot E = 0 \tag{7}$$

$$\nabla \cdot B = 0 \tag{8}$$

Note:

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) \tag{9}$$

$$= B(A \cdot C) - (A \cdot B)C \tag{10}$$

If $f(x,t)$ is real valued and continuous, then (Schwarz-Clairaut-Young Theorem):

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial t} f(x,t) = \frac{\partial}{\partial t}\frac{\partial}{\partial x_i} f(x,t) \tag{11}$$

Now, consider:

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{\partial B}{\partial t}\right) \tag{12}$$

By (10):

$$\nabla(\nabla \cdot E) - (\nabla \cdot \nabla)E = \nabla \times \left(-\frac{\partial B}{\partial t}\right) \tag{13}$$

With (7):

$$-(\nabla \cdot \nabla)E = \nabla \times \left(-\frac{\partial B}{\partial t}\right) \tag{14}$$

Taking out the $-$, and noting $\nabla \cdot \nabla = \nabla^2$:

$$\nabla^2 E = \nabla \times \left(\frac{\partial B}{\partial t}\right) \tag{15}$$

Using (11):

$$\nabla^2 E = \frac{\partial}{\partial t}(\nabla \times B) \tag{16}$$

Using (6):

$$\boxed{\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}} \tag{17}$$

Which is the wave equation for a (electromagnetic) wave with velocity c.