Exposition on Lorentz transforms in special relativity

In a frame, the position is described by the position 4-vector:

$$\vec{x} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1}$$

Let there be two frames, f and f', and let an observer at the origin of frame f' see the origin of frame f moving away at velocity u along the x axis. Then, if the origins and orientations of the axes correspond at time zero, to convert the position 4-vector from frame f to frame f', use the Lorentz transform matrix L:

$$\vec{x}\,' = L(u)\,\vec{x} \tag{2}$$

Where:

$$L = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

with $\beta = u/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

Note that:

$$L^{-1} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

So:

$$\vec{x} = L^{-1}\vec{x}\,' = L^{-1}(L\,\vec{x}) \tag{5}$$

Example 1: Decay of Muons

In a muon's rest frame it has a half-life of 2.2μs. If we look at a muon and see it traveling at a velocity of $0.95c$, and in the muon's frame it decays after 2.2μs, how far does it travel in our frame before decaying? How long does it live?

Here $\vec{x} = (ct, 0, 0, 0)^\top$, so:

$$\vec{x}\,' = L\vec{x} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct \\ \beta\gamma ct \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

Here $\beta = 0.95$ and $\gamma = 3.203$, so:

$$\vec{x}\,' = L\vec{x} = \begin{pmatrix} 3.203ct \\ 3.042ct \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (7.046\ \mu\text{s})c \\ 2008\ \text{m} \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

so it travels 2008 meters in 7.046 μs in our frame.

Example 2: Two spaceships

Suppose one spaceship sees a second spaceship approaching at $u = (2/3)c$, and the second space ship then sends out a messsenger craft towards the first ship at $v = (1/2)c$ in the second ship's frame. How fast is the messenger approaching in the first ship's frame?

Here the 4-position of the messenger in the second frame is $\vec{x}_2 = (ct, ct/2, 0, 0)^\top$, so for the 4-position of the messenger in the first frame is:

$$\vec{x}_1 = L\vec{x}_2 = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ ct/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct + \beta\gamma ct/2 \\ \beta\gamma ct + \gamma ct/2 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

Here $\beta = 2/3$ and $\gamma = 1/\sqrt{1-(2/3)^2} = 3/\sqrt{5}$, so:

$$\vec{x}_1 = \begin{pmatrix} (8/2\sqrt{5})ct \\ (7/2\sqrt{5})ct \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Now, in the first frame, the messenger's velocity is:

$$\text{velocity}_1 = \frac{\Delta x_1}{\Delta t_1} = \frac{(7/2\sqrt{5})ct}{(8/2\sqrt{5})t} = \frac{7}{8}c \tag{10}$$

Example 3: Angle of a laser

Suppose that an astronaut on a spaceship passes passes by an observer on an asteroid with velocity v relative to the asteroid. In the astronaut's reference frame they shine a beam of laser light at an angle θ from the horizontal. What is this angle in the reference frame?

We write the position of the end of the light beam as a 4-vector in the astronaut's frame: $\vec{x}_{\text{beam}} = (ct, \cos(\theta)ct, \sin(\theta)ct, 0)$, so the end of the light beam in the observer's frame is at:

$$\vec{x}'_{\text{beam}} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ \cos(\theta)ct \\ \sin(\theta)ct \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct + \beta\gamma\cos(\theta)ct \\ \beta\gamma ct + \gamma\cos(\theta)ct \\ \sin(\theta)ct \\ 0 \end{pmatrix} \tag{11}$$

Now, we must adjust for the position of the origin in the astronaut's frame as perceived in the observer's frame, which is:

$$\vec{x}'_{\text{origin}} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct \\ \beta\gamma ct \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

Therefore, in the observer's frame, they see:

$$\vec{x}'_{\text{beam}} - \vec{x}'_{\text{origin}} = \begin{pmatrix} \beta\gamma\cos(\theta)ct \\ \gamma\cos(\theta)ct \\ \sin(\theta)ct \\ 0 \end{pmatrix} \tag{13}$$

whose $\hat{x}$ component is $\gamma\cos(\theta)ct$ and whose $\hat{y}$ component is $\sin(\theta)ct$, so the angle is:

$$\theta' = \tan^{-1}\left(\frac{\sin(\theta)ct}{\gamma\cos(\theta)ct}\right) = \tan^{-1}\left(\tan(\theta)/\gamma\right) \tag{14}$$

Exercise: A line of space ships

Suppose that an observer on earth sees a space ship moving away at $v_1 = c/2$, an observer on the space ship sees a space ship moving away at $v_2 = c/2$, an observer on the second space ship sees a space ship moving away at $v_3 = c/2$, and so on. Show that the velocity of the nth ship as perceived by an observer on earth is:

$$v_n = \frac{3^n - 1}{3^n + 1} c \tag{15}$$