Stationary Points in the Plane for Regular Polygons of Fixed Point Charges

Spenser Talkington, November 10, 2019

The Question

Suppose that we have $n \geq 3$ point charges, each with charge q, and we fix them in place at the corners of a regular polygon with n sides and apothem a. Now, if we have an additional charge Q, that is free to move, and we place it at rest in the plane, at what points will it stay at rest? Assume that the electric force is the only relevant force, that $\boldsymbol{F} = q\boldsymbol{E}$, and:

$$\boldsymbol{E} = \frac{kq}{r^2}\,\hat{\boldsymbol{r}} \tag{1}$$

The solution to is simple: Q will stay at rest when the electric field is 0:

$$\boldsymbol{E} = 0 \tag{2}$$

One solution is the center of the polygon. However, perhaps counterintuitively, n more solutions exist. The purpose of this report is to build intuition for the the behavior of charges in an electric field.

Ansatz

We propose that there is a family of solutions in addition the point at the center: saddle points of the scalar potential, which lie on the n apothems. In polar coordinates, these points are located at $\theta = 2\pi(i + 1/2)/n$ for $i \in \{0, 1, \ldots, n-1\}$, where:

$$r_n \in (0, a) \tag{3}$$

This ansatz is sensible because to first order, the scalar potential in between any two neighboring point charges on the polygon is the scalar potential of two point equal charges. For this potential, we know there is a saddle point equally spaced between the points. Higher order corrections move the saddle off the side of the polygon and give the precise location of the stationary point. Note the saddle point is an unstable equilibrium for both $Q < 0$ and $Q > 0$, thus the need to neglect gravity and other forces.

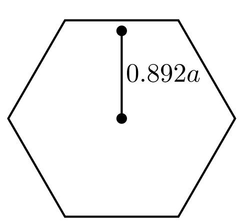

FIG. 1. Two of the seven stationary points in a hexagon. Drawn to scale. Note that one point is near, but not at the center of a side.

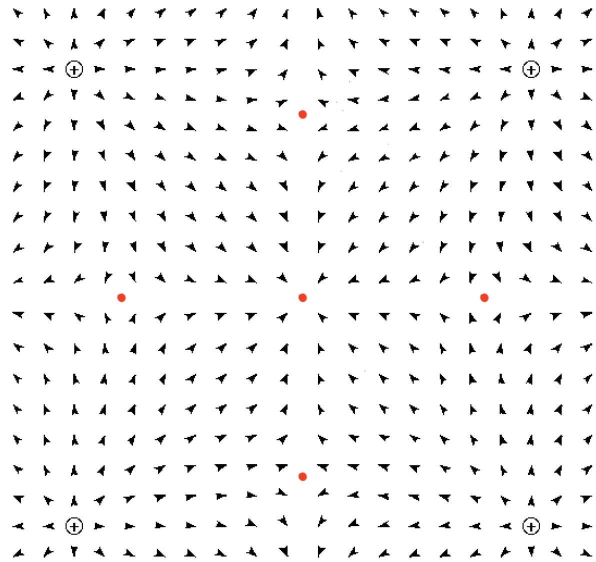

FIG. 2. Direction of the electric field for an square of charges. Stationary points are marked in red.

Solution for Radii

Zeros of the Electric Field

For n small it is possible to write the field explicitly and unambiguously, while for larger n, the direction of contribution to the electric field by a given point charge becomes ambiguous. For a square of charges, we equate the forces from equation 1 along the apothem, with $kq = 1$, $a = 1$, and $r \in (-1, 1)$:

$$\frac{1+r}{\left((1+r)^2+1\right)^{3/2}} - \frac{1-r}{\left((1-r)^2+1\right)^{3/2}} = 0 \tag{4}$$

For a hexagon, the equation is:

$$\frac{r}{\left(r^2+\frac{4}{3}\right)^{3/2}} + \frac{1+r}{\left((1+r)^2+\frac{1}{3}\right)^{3/2}} - \frac{1-r}{\left((1-r)^2+\frac{1}{3}\right)^{3/2}} = 0 \tag{5}$$

The solutions to these equations are most clearly expressed numerically, where the radii for a square are $r_4 \in \{0, \pm 0.773485\}$, and the radii for a hexagon are $r_6 \in \{0, \pm 0.892333\}$.

While this method works for a few charges, it breaks down, even at $n = 5$. It is equivalent and unambiguous to find the critical points of the scalar potential.

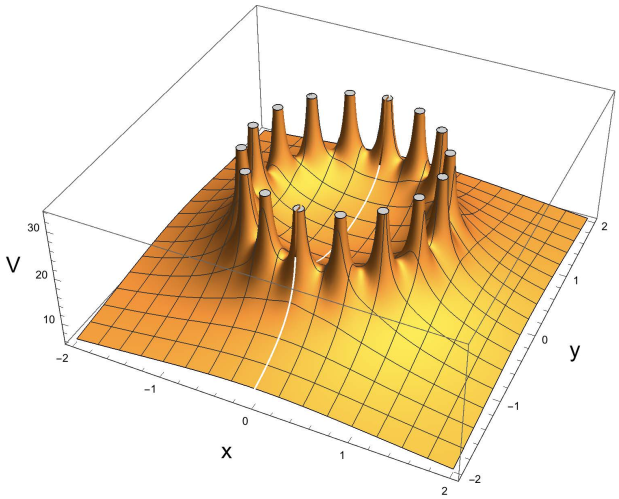

FIG. 3. Scalar potential for $n = 16$ charges. Note the local minimum at $(0,0)^T$, and the saddle points near the edges of the polygon.

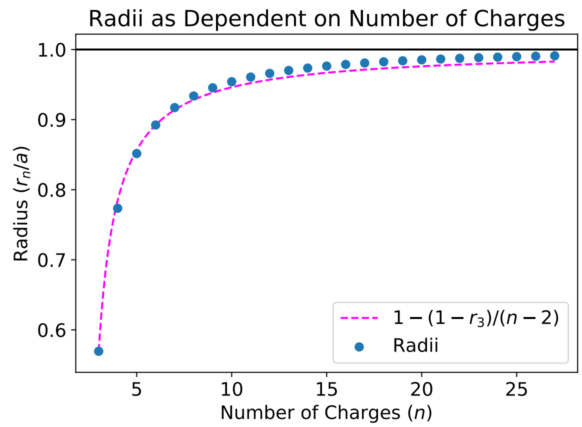

FIG. 4. Radii asymptotically approach 1 for large n. The radii can be approximately interpolated by a function of form $1 - 1/n$ that diverges at $n = 2$ and interpolates $(3, r_3)$.

Critical Points of the Scalar Potential

The electric field is the negative gradient of the scalar potential, so zeros of the electric field are critical points of the scalar potential:

$$\boldsymbol{E} = -\nabla V \tag{6}$$

For simplicity, we express the scalar potential in polar coordinates, where R is the circumradius $a/\cos(\pi/n)$:

$$V(r,\theta) = kq \sum_{i=0}^{n-1} \frac{1}{\sqrt{R^2 + r^2 - 2Rr\cos(2\pi i/n - \theta)}} \tag{7}$$

We then proceed numerically:

1. Express $V(r,\theta)$ as $V(x,y)$ using $r = \sqrt{x^2+y^2}$, and $\theta = \arctan(y/x)$
2. Find $\boldsymbol{E}$, but don't project onto the apothem
3. Find $\boldsymbol{E}$'s zeros using Mathematica's `FindRoot`
4. Calculate the radius

Backstory

I first thought of a version of this problem when I posed a question to my electromagnetism students:

"Given a square of side length a with charges q fixed on each corner, describe the subsequent motion for a charge q released at rest anywhere in the square. I.e. does the charge stay in the square, move to the center, move towards a fixed charge, head to infinity, oscillate, etc."

When I plotted the field as in Fig. 2, I realized there were two regions: one where the charge goes to infinity, one where the charge oscillates, and five points where the charge stays at rest.

I was then curious as to the location of these stationary points, so I investigated. While the equations for the radii are algebraic, their solutions are not simple, so I proceeded numerically. I selected Mathematica to implement a solution simply, and to visualize the potential easily.

I have become aware that this problem has been considered to some extent at Physics StackExchange.