In these notes we review coherent, squeezed, multi-mode, entangled, and thermal states in quantum optics. We show how to decompose into number states.

Classical references are Phys. Rev. A 13, 2226 (1976) and Phys. Rev. D 23, 1693 (1981), and modern reviews are Rev. Mod. Phys. 84, 621 (2012) and Rev. Mod. Phys. 92, 035005 (2020).

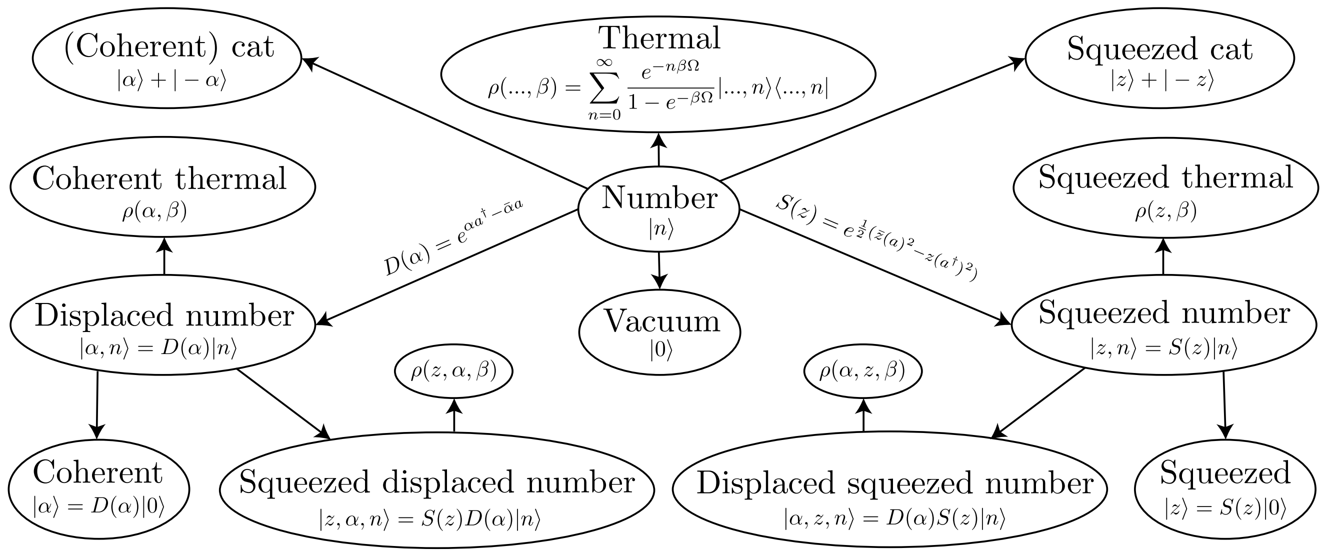

Fig. 1 Types of single-mode states in quantum optics generated by displacement $D(\alpha)$, squeezing $S(z)$, and their corresponding thermal states. The number states $|n\rangle$ (Fock states) form a complete and orthonormal basis, and we present some expressions for the states listed above in terms of the number states.

Single Mode

Starting with the number state/Fock state we can generate other states through displacement $D(\alpha)$ and squeezing $S(z)$:

$$D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha} a}$$
$$S(z) = e^{\frac{1}{2}(\bar{z}(a)^2 - z(a^\dagger)^2)}$$

where α and z are complex numbers and $\bar{z}$ is the complex conjugate of z.

Let us list the various types of photonic states we might encounter:

1. **Number state/Fock state**. This is our staring point:

$$|n\rangle = |n\rangle$$

2. **Displaced number state**

$$\begin{aligned} |\alpha, n\rangle &= D(\alpha)|n\rangle \\ &= \sum_{m=0}^{\infty} |m\rangle\langle m|D(\alpha)|n\rangle \\ &= \sum_{m=0}^{\infty} \left(\sqrt{\frac{n!}{m!}} \alpha^{m-n} e^{-|\alpha|^2/2} L_n^{(m-n)}(|\alpha|^2) \right) |m\rangle \end{aligned}$$

where we used the relation from Physical Review 177, 1857 (1969) for generalized Laguerre polynomials L

$$\langle m|D(\alpha)|n\rangle = \sqrt{\frac{n!}{m!}}\alpha^{m-n}e^{-|\alpha|^2/2}L_n^{(m-n)}(|\alpha|^2)$$

3. **Coherent state**. The coherent state is the displaced $n=0$ state

$$\begin{aligned}|\alpha\rangle &= |\alpha,0\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^m}{\sqrt{m!}}e^{-|\alpha|^2/2}L_0^{(m)}(|\alpha|^2)\right)|m\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^m}{\sqrt{m!}}e^{-|\alpha|^2/2}\right)|m\rangle\end{aligned}$$

where we used that $L_0^{(m)} = 1$ for all m.

4. **Coherent cat state**. The coherent cat state is

$$\begin{aligned}|\text{cat}\rangle &= |\alpha\rangle + |-\alpha\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^m}{\sqrt{m!}}e^{-|\alpha|^2/2}\right)|m\rangle + \sum_{m=0}^{\infty}\left(\frac{(-\alpha)^m}{\sqrt{m!}}e^{-|\alpha|^2/2}\right)|m\rangle \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^m+(-\alpha)^m}{\sqrt{m!}}e^{-|\alpha|^2/2}\right)|m\rangle\end{aligned}$$

5. **Squeezed number state**

$$\begin{aligned}|z,n\rangle &= S(z)|n\rangle \\ &= \sum_{m=0}^{\infty}|m\rangle\langle m|S(z)|n\rangle \\ &= \sum_{m=0}^{\infty}e^{i(n-m)\theta/2}G_{mn}(r)|m\rangle\end{aligned}$$

where we introduced $z = re^{i\theta}$ and $G_{mn} = \langle m|S(z)|n\rangle$ whose coefficients are (Phys. Rev. D 35, 400 (1985))

$$G_{mn} = \begin{cases}(-1)^{m+n/2}\sqrt{\frac{m!n!}{\cosh(r)}}\left(\frac{\tanh(r)}{2}\right)^{(m+n)/2}\sum_l \frac{(-4/\sinh^2(r))^l}{(2l)!(m/2-l)!(n/2-l)!} & m,n \text{ both even} \\ (-1)^{m+n/2-3/2}\sqrt{\frac{m!n!}{\cosh^3(r)}}\left(\frac{\tanh(r)}{2}\right)^{(m+n)/2-1}\sum_l \frac{(-4/\sinh^2(r))^l}{(2l+1)!((m-1)/2-l)!((n-1)/2-l)!} & m,n \text{ both odd} \\ 0 & \text{otherwise}\end{cases}$$

6. **Squeezed state**

$$\begin{aligned}|z\rangle &= |z,0\rangle \\ &= \sum_{m=0}^{\infty}\frac{\sqrt{(2m)!}}{2^m m!}\frac{(-e^{i\theta}\tanh(r))^m}{\sqrt{\cosh(r)}}|2m\rangle\end{aligned}$$

7. **Squeezed cat state**

$$
\begin{aligned}
|\text{s-cat}\rangle &= |z\rangle + |-z\rangle \\
&= \sum_{m=0}^{\infty} \frac{\sqrt{(2m)!}}{2^m m!} \frac{(-1)^m [e^{im\theta} + e^{-im\theta}] \tanh^m(r)}{\sqrt{\cosh(r)}} |2m\rangle \\
&= \sum_{m=0}^{\infty} \cos(m\theta) \frac{\sqrt{(2m)!}}{2^{m-1} m!} \frac{(-1)^m \tanh^m(r)}{\sqrt{\cosh(r)}} |2m\rangle
\end{aligned}
$$

8. **Thermal states**. We can turn any of the states with an n dependence into a thermal state through the summation

$$\rho(\ldots, \beta) = \sum_{n=0}^{\infty} \frac{e^{-n\beta\Omega}}{1 - e^{-\beta\Omega}} |\ldots, n\rangle\langle \ldots, n|$$

where β is the inverse temperature. For example the "thermal state" is

$$\rho(\beta) = \sum_{n=0}^{\infty} \frac{e^{-n\beta\Omega}}{1 - e^{-\beta\Omega}} |n\rangle\langle n|$$

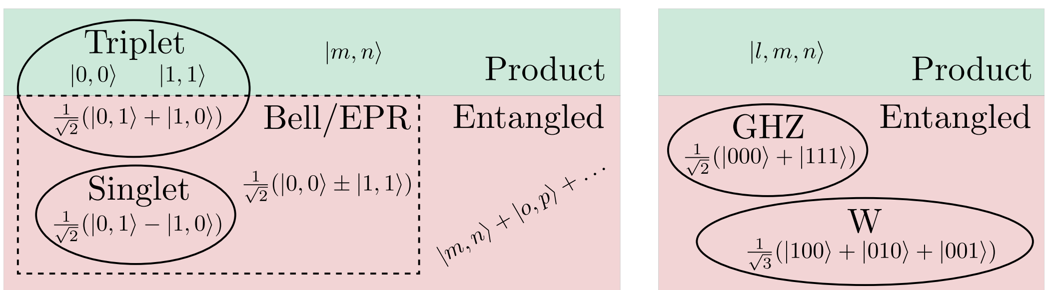

Fig. 2 Types of (pure/non-mixed) multi-mode states in quantum optics; these can be divided into product states and entangled states. Here we list archetypal states that find common use in the literature and do not classify systematically. Left: two-mode states. Right: three-mode states.

Multi-Mode, Polarization, and Entanglement

We may also consider multimode systems with Fock states $|l, m, n, \ldots\rangle$ where there are l photons in mode 1, m photons in mode 2, n photons in mode 3, etc. These modes may be different from each other, for example by having different frequencies Ω_1, Ω_2, Ω_3, or by corresponding to different polarization states. Consider for example x and y polarization states $|n_x, n_y\rangle$. In this case the state $\frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle)$ corresponds to a Bell/EPR state of two photons that are either both x polarized or both y polarized. We illustrate some common states in Fig. 2:

1. **Singlet and Triplet states** (2 modes): "spin-0" and "spin-1" combinations of two modes

2. **Bell/EPR pair states** (2 modes): maximally entangled two-mode states

3. **GHZ state** (3 modes): entangled 3-mode state (fragile to measurement of one mode)

4. **W state** (3 modes): entangled 3-mode state (robust to measurement of one mode)