On the Positive P Representation

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P Representation

- Glauber and Sudarshan
- Key idea: (over)complete basis can represent ρ

$$\rho = \int d\bar{\alpha} d\alpha \; e^{-\bar{\alpha}\alpha} |\alpha\rangle\langle\alpha| \, P(\bar{\alpha}, \alpha)$$

- Converts second quantized operators to differential operators (PDE)

$$\begin{aligned} a\rho &\to \alpha P \\ a^\dagger \rho &\to (\bar{\alpha} - \partial_\alpha) P \\ \rho a &\to (\alpha - \partial_{\bar{\alpha}}) P \\ \rho a^\dagger &\to \bar{\alpha} P \end{aligned}$$

etc, etc

- *Sometimes* phase space PDE is easy to solve (some Fokker-Planck)

Positive P Representation

- Sometimes the P Representation leads to a hard to solve PDE or a PDE with un-physical solutions
- Drummond and Gardiner, J. Phys. A 13, 2353 (1980)

$$\hat{\rho} = \int P(\alpha, \beta)\hat{\Lambda}(\alpha, \beta)\, \mathrm{d}\mu\ (\alpha, \beta).$$

$$\hat{\Lambda}(\alpha, \beta) = |\alpha\rangle\langle\beta^*|/(\langle\beta^*|\alpha\rangle) = \exp(\alpha\hat{a}^+ - \alpha\beta)|0\rangle\langle 0|\exp(\beta\hat{a})$$

P $\quad \mathrm{d}\mu(\alpha, \beta) = \delta^2(\alpha^* - \beta)\, \mathrm{d}^2\alpha\, \mathrm{d}^2\beta.$

Positive P $\quad \mathrm{d}\mu(\alpha, \beta) = \mathrm{d}^2\alpha\, \mathrm{d}^2\beta.$

- Correspondences

$$\hat{a}\hat{\Lambda}(\boldsymbol{\alpha}) = \alpha\hat{\Lambda}(\boldsymbol{\alpha})$$

$$\hat{a}^+\hat{\Lambda}(\boldsymbol{\alpha}) = (\beta + \partial/\partial\alpha)\hat{\Lambda}(\boldsymbol{\alpha})$$

$$\hat{\Lambda}(\boldsymbol{\alpha})\hat{a}^+ = \hat{\Lambda}(\boldsymbol{\alpha})\beta$$

$$\hat{\Lambda}(\boldsymbol{\alpha})\hat{a} = (\partial/\partial\beta + \alpha)\hat{\Lambda}(\boldsymbol{\alpha}).$$

Phase space representations

- P (Glauber-Sudarshan): normal ordered
- Wigner: symmetrically ordered
- Q (Husimi): anti-normal ordered

- Then can have different representations by integration measure
  - “diagonal”
  - “generalized” (complex)
  - “generalized” (positive)

Literature survey: Second Harmonic Generation

- Key realization: we are essentially considering cavity second harmonic generation! This has been treated by several authors
  - Drummond, McNeil, Walls, Optica Acta 28, 211 (1981)
  - Dorfle and Schenzle, Z. Phys. B 65, 113 (1986)
  - G.S. Holliday Thesis U. Arkansas (1989)
  - McNeil and Craig, PRA 41, 4009 (1990)

Drummond, McNeil, Walls, Optica Acta 28, 211 (1981)

$$\frac{\partial}{\partial t}\rho = \frac{1}{i\hbar}[H_{\text{rev}}, \rho] + \gamma_1(2a_1\rho a_1^\dagger - a_1^\dagger a_1\rho - \rho a_1^\dagger a_1) + \gamma_2(2a_2\rho a_2^\dagger - a_2^\dagger a_2\rho - \rho a_2^\dagger a_2)$$

$$H_{\text{rev}} = \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_2 a_2^\dagger a_2 + i\hbar\frac{\kappa}{2}(a_1^{\dagger 2}a_2 - a_1^2 a_2^\dagger) + i\hbar(\varepsilon_1 a_1^\dagger \exp(-i\omega_{\text{p}}t) - \varepsilon_1^* a_1 \exp(i\omega_{\text{p}}t)) + i\hbar(\varepsilon_2 a_2^\dagger \exp(-2i\omega_{\text{p}}t) - \varepsilon_2^* a_2 \exp(2i\omega_{\text{p}}t)),$$

$$\frac{\partial}{\partial t}P(\boldsymbol{\alpha}) = \Bigg\{ \frac{\partial}{\partial\alpha_1}[(\gamma_1 + i\Delta_1)\alpha_1 - \varepsilon_1 - \kappa\alpha_1^\dagger\alpha_2] + \frac{\partial}{\partial\alpha_1^\dagger}[(\gamma_1 - i\Delta_1)\alpha_1^\dagger - \varepsilon_1^* - \kappa\alpha_1\alpha_2^\dagger] + \frac{\partial}{\partial\alpha_2}\left[(\gamma_2 + i\Delta_2)\alpha_2 - \varepsilon_2 + \frac{\kappa}{2}\alpha_1^2\right] + \frac{\partial}{\partial\alpha_2^\dagger}\left[(\gamma_2 - i\Delta_2)\alpha_2^\dagger - \varepsilon_2^* + \frac{\kappa}{2}\alpha_1^{\dagger 2}\right] + \frac{1}{2}\left[\frac{\partial^2}{\partial\alpha_1^2}(\kappa\alpha_2) + \frac{\partial^2}{\partial\alpha_1^{\dagger 2}}(\kappa\alpha_2^\dagger) + \Gamma_1\frac{\partial^2}{\partial\alpha_1\partial\alpha_1^\dagger} + \Gamma_2\frac{\partial^2}{\partial\alpha_2\partial\alpha_2^\dagger}\right]\Bigg\} P(\boldsymbol{\alpha}),$$

$$\Delta_1 = \omega_1 - \omega_{\text{p}}, \Delta_2 = \omega_2 - 2\omega_{\text{p}} \qquad \alpha_1 \to \alpha_1 \exp(-i\omega_{\text{p}}t), \quad \alpha_2 \to \alpha_2 \exp(-2i\omega_{\text{p}}t).$$

Drummond, McNeil, Walls, Optica Acta 28, 211 (1981)

1. Linearized (approximate) solution to equation
2. Exact steady-state solution for $\gamma_2 \gg \gamma_1, \Gamma_2 = 0$
   - Physical for us—Higgs relaxes fast
   - Key idea: reduces to equation for just a_1 with additional terms
   - Large Γ_1 solution (thermally dominated)

$$P(\alpha_1, \alpha_1^\dagger) = \mathcal{N} \exp\left[-\Phi(\alpha_1, \alpha_1^\dagger)\right], \qquad \Phi(\alpha_1, \alpha_1^\dagger) = \frac{1}{\Gamma_1}\left\{ \frac{\kappa^2}{4\gamma_2}(\alpha_1^\dagger \alpha_1)^2 + \gamma_1 \alpha_1^\dagger \alpha_1 - \frac{\kappa \varepsilon_2^*}{2\gamma_2}\alpha_1^2 - \frac{\kappa \varepsilon_2}{2\gamma_2}\alpha_1^{\dagger 2} - \varepsilon_1^* \alpha_1 - \varepsilon_1 \alpha_1^\dagger \right\}$$

   - Small Γ_1 solution (“quantum” dominated)

$$P(\alpha_1, \alpha_1^\dagger) = \mathcal{N} \exp\left[2\alpha_1^\dagger \alpha_1 + \frac{2\bar{\gamma}_1 \gamma_2}{\kappa^2} \ln(c^2 - \kappa^2 \alpha_1^2) + 2\left(\frac{\bar{\gamma}_1 \gamma_2}{\kappa^2}\right)^* \ln(c^{*2} - \kappa^2 \alpha_1^{\dagger 2}) + \frac{2\gamma_2 \varepsilon_1}{c\kappa} \ln\left(\frac{c + \kappa\alpha_1}{c - \kappa\alpha_1}\right) + 2\left(\frac{\gamma_2 \varepsilon_1}{c\kappa}\right)^* \ln\left(\frac{c^* + \kappa\alpha_1^\dagger}{c^* - \kappa\alpha_1^\dagger}\right)\right]$$

G.S. Holliday Thesis U. Arkansas (1989)

- Same limit as Drummond, McNeil and Walls (fast decay of ω_H)
  - 3.11, 4.8, and 4.16 give positive P representation and exact solution
- Focus on anti-bunching! Experiment too

McNeil and Craig, PRA 41, 4009 (1990)

1. Numeric study (instabilities, etc)
2. Analytic solution (no driving, initial coherent state), $\gamma_2 \gg \gamma_1$

$$\alpha_1 = \frac{\alpha_1(0) e^{-(\nu - \frac{1}{2})t + i w_1}}{\left[1 + 2\alpha_1(0)^2 \int_0^t dt\, e^{-2(\nu - \frac{1}{2})t + i(w_1 - w_2)}\right]^{\frac{1}{2}}}$$

$$\beta_1 = \alpha_1 e^{-i(w_1 + w_2)},$$

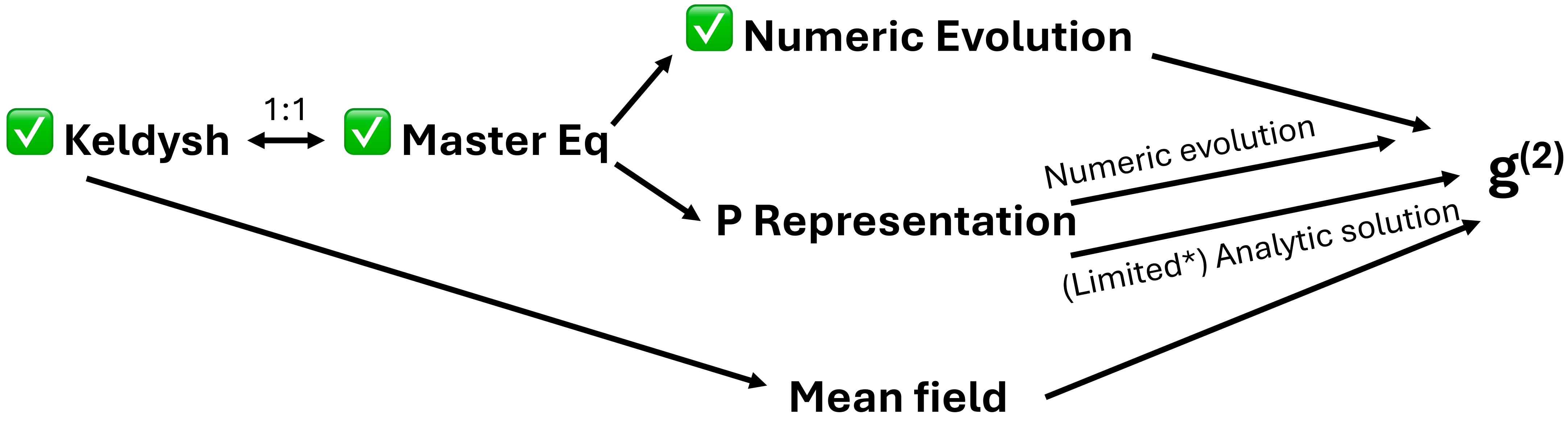
Dorfle and Schenzle, Z. Phys. B 65, 113 (1986)

1. Numeric study (attractors, bifurcations, limit cycles)
2. Analytic solution: cavity resonant with driving

$$\alpha_1(t) = \mathrm{i}\,\varepsilon\, c_{11} \cos(\Omega t + \varphi_{11}) + \varepsilon^2 c_{12} \cos(\Omega t + \varphi_{12})$$
$$\alpha_2(t) = \mathrm{i}\,\varepsilon\, c_{21} \cos(\Omega t + \varphi_{21}) + \varepsilon^2 c_{22} \cos(\Omega^t + \varphi_{22})$$

- See also Mandel and Erneux, Optica Acta 28, 132 (1982)

Summary



* 1. Resonant driving
2. Fast decay of Higgs