Abstract

In these notes we introduce thermalization and many body localization (MBL) through the specific example of the XXZ model with a random transverse and a fixed parallel field. We then conduct a moderate-size exact diagonalization study of level spacing statistics with up to $L = 16$ sites in the half-filling particle sector, diagonalizing matrices up to size 12870. We reproduce the results of PRB **75**, 155111 (2007). Finally, we review the recent controversy on the (non)-existence of MBL and provide further reading.

Thermalization and Localization

A quantum particle in an infinite well is localized, but a particle in any finite well has a finite tunneling probability to other wells. A priori, quantum tunneling would seem to dictate that the particle explore physical and parameter space. Except for in finely tuned integrable systems where an extensive number of locally conserved quantities enable trajectories to close we expect quantum systems to thermalize. Thermalization is the process where time evolution brings the density matrix arbitrarily close to a thermal density matrix at long times. While the time evolution may be unitary and the system remains in a pure state ergodicity, or the exploration of phase space, leads to thermalization. This is the eigenstate thermalization hypothesis (ETH) that in all but a few closed-orbit systems information will end up in inaccessable non-local degrees of freedom at long times.

Thermalization is challenged by localization where the destructive interference of phases leads to the (Anderson) localization of information at long times. Generalizing to many-body systems, disorder is postulated to localize information even at long times.

Poisson and Wigner-Dyson Statistics

In thermalized, diffusive systems one expects random matrix theory statistics for the distribution of eigenvalues, e.g. GOE statistics, where no eigenvalues are degenerate. In localized systems one expects Poissonian statistics where eigenvalues can bunch. A more robust statistical measure is the level spacing ratio

$$r_n = \frac{\min(E_{n+1} - E_n, E_{n+2} - E_{n+1})}{\max(E_{n+1} - E_n, E_{n+2} - E_{n+1})}$$

XXZ Model With Two Fields

Let us consider the XXZ spin model with L sites and open boundary conditions

$$H_{XXZ} = J \sum_{i=1}^{L-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_z \sum_{i=1}^{L-1} S_i^z S_{i+1}^z.$$

Now this model is integrable, so system dynamics will form closed trajectories and the system will retain knowledge of its initial state at long times. To break integrability we add a long-ranged parallel field term

$$H_{\parallel} = h_x \sum_{i=1}^{L} S_i^x,$$

where the Jordan-Wigner string breaks the locally conserved quantities, which will open the trajectories and result in diffusive behavior where the system state at long times will retain little information about the initial state. To compete with this we add random local potentials h_i and seek to numerically answer whether localization occurs

$$H_{\perp} = \sum_{i=1}^{L} h_i S_i^z,$$

so that we have $H = H_{XXZ} + H_{\parallel} + H_{\perp}$. Now we can express this in terms of fermions using the Jordan-Wigner transformation

$$\begin{aligned} S_i^x &= \tfrac{1}{2}(S_i^+ + S_i^-) \\ S_i^y &= \tfrac{1}{2i}(S_i^+ - S_i^-) \\ S_i^+ &= e^{-i\pi \sum_{j<i} c_j^\dagger c_j} c_i^\dagger \\ S_i^- &= e^{i\pi \sum_{j<i} c_j^\dagger c_j} c_i \\ S_i^z &= c_i^\dagger c_i - \tfrac{1}{2}, \end{aligned}$$

so that the Hamiltonian is

$$\begin{aligned} H_{XXZ} &= \frac{J}{2} \sum_{i=1}^{L-1} c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + J_z \sum_{i=1}^{L-1} (n_i - \tfrac{1}{2})(n_{i+1} - \tfrac{1}{2}) \\ H_{\parallel} &= \frac{h_x}{2} \sum_{i=1}^{L} e^{-i\pi \sum_{j<i} n_j} c_i^\dagger + e^{i\pi \sum_{j<i} n_j} c_i \\ H_{\perp} &= \sum_{i=1}^{L} h_i (n_i - \tfrac{1}{2}). \end{aligned}$$

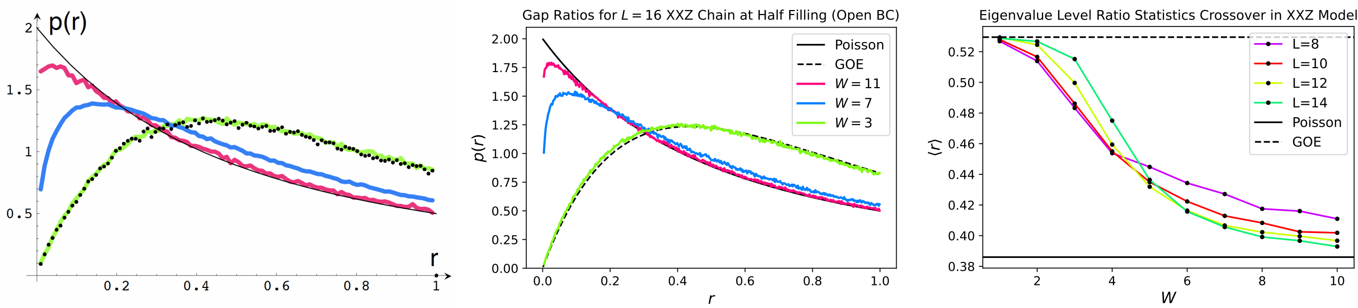

Fig. Left: as disorder strength increases the system transitions from diffusive GOE level ratio statistics to localized Poisson statistics (from PRB **75**, 155111 (2007)). Middle: My reproduction of the results using exact diagonalization. Right: There is a crossover in the level ratio statistics with disorder strength.

Numeric Methods

Now the model presented violates particle number conservation by the lone creation and annihilation operators in $H_{\parallel}$. To remedy this, but to still study the same essential physics, we can replace this term with another term that breaks integrability. One such term is a next nearest neighbor hopping term

$$H_{NNN} = \frac{J'}{2} \sum_{i=1}^{L-2} c_i^\dagger c_{i+2} + c_{i+2}^\dagger c_i$$

which importantly preserves particle number. Doing this and taking $H_\perp \mapsto \sum_{i=1}^{L} h_i n_i$ which corresponds to changing the disorder distribution, we arrive at the Hamiltonian studied by Oganesayan and Huse in PRB **75**, 155111 (2007). A naïve implementation in Python may take a day to average over 100 trials at $L = 12$. We can do much better than this.

The new Hamiltonian conserves particle number which allows us to project it into the fixed particle number sector. This is roughly the equivalent of adding two sites for no more computational cost. Now we are interested in the average properties over many disorder realizations, and the only thing that changes between trials are the h_i. This enables us to cache H_{XXZ} and H_{NNN} as well as $\{n_i\}$ rather than recalculating them with each trial. Caching enables us to study about two more sites. This brings us up to $L = 16$. Now recoding this in C++, using a computer 100x more powerful, and waiting a month could be used to get about six more sites, bringing us up to $L = 22$ which is close to the state of the art $L = 24$.

Now we study this model with open boundaries and $J = J_z = J' = 2$ where h_i are chosen from a normal distribution with mean 0 and standard deviation W. We average the data over 400 disorder realizations.

Does MBL Exist?

While analytical arguments suggest that many-body localization should exist and that there should be a crossover between random matrix and Poisson statistics, the numeric evidence at larger system sizes comes into conflict with this. As we see by looking at the crossover of the level ratio statistics, there different system sizes crossover at disorder strengths that increase with system lengths. This lead Oganesayan and Huse to conclude that “based on spectral statistics alone, we have thus been unable to make a strong numerical case for the presence of a many-body localized phase.”

Subsequently there was great interest in the ETH to MBL phase transition, but numeric limitations plagued the search for the transition and exploration of its properties (if it exists). This and larger scale numeric studies in PRE **102**, 062144 (2020) and PRB **104**, 201117 (2021) have inspired a crisis in faith as to whether such a transition or localized phase even exists. Are numerics wrong again and MBL is vindicated in the thermodynamic limit, or do analytic arguments for MBL fail somewhere?

Further Reading

I based most of my project on PRB **75**, 155111 (2007) and PRB **47**, 11487 (1993). Interesting alternative persepectives on localization are given by entanglement entropy as in PRB **77**, 064426 (2008) and PRL **109**, 017202 (2012), and by the sensitivity of systems to boundary conditions (Thouless energy) PRB **82**, 174411 (2010). Experiments claiming to observe MBL include Science **352**, 1547 (2016) and Science **364**, 256 (2019). Ann. Rev. CMP **6**, 15 (2015) and RMP **91**, 021001 (2019) are pedagogical reviews.