Effective Action for Cavity Input-Output Theory

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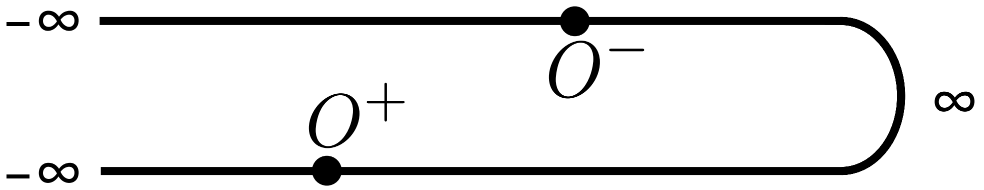
Keldysh Contour

- Take expectation values at times on the Keldysh contour



- Quick recap
  - Two contours because time evolution from the state we know $\rho=|\psi(-\infty)\rangle\langle\psi(-\infty)|$ to where we want at time t is a similarity transtormation $\rho(t) = U\,\rho(-\infty)\,U^+$
  - Plus and minus fields because we only want to do one integral $-\infty$ to ∞
  - Bar and non-bar fields because we want to create/annihilate coherent states
    - Coherent states are eigenstates of b and b^+: $b|\psi\rangle= \psi|\psi\rangle$ and $\langle\psi|b^+=\langle\psi|\bar{\psi}$

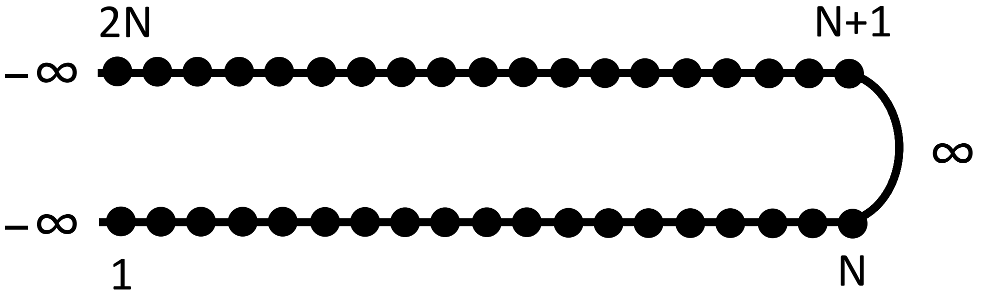
Discretized Action

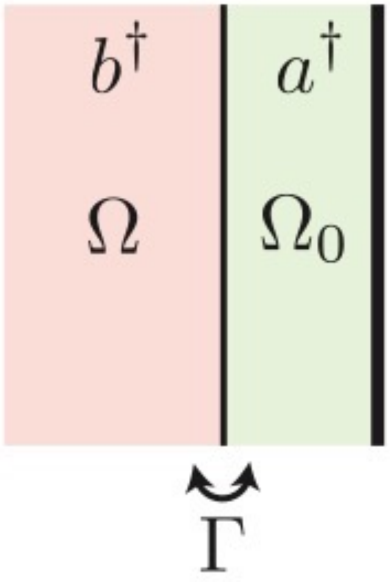
- Consider the action for time slices i=1,…N on the forward contour, N+1,…2N on the reverse contour

$$S_{\text{tot}} = S_A + S_B + S_{AB}$$

$$S_B[b^*, b] = \sum_{\Omega} \left(i b^*_{1,\Omega} b_{1,\Omega} - i\rho(\Omega) b^*_{1,\Omega} b_{2N,\Omega} + \sum_{i=2}^{2N} i b^*_{i,\Omega} b_{i,\Omega} - i b^*_{i,\Omega} b_{i-1,\Omega} - \Omega \delta t b^*_{i,\Omega} b_{i-1,\Omega} \right)$$

$$S_{AB}[a^*, a, b^*, b] = \sum_{\Omega} \sum_{i=2}^{2N} -\delta t (\gamma_\Omega a^*_i b_{i-1,\Omega} + \gamma^*_\Omega b^*_{i,\Omega} a_{i-1})$$

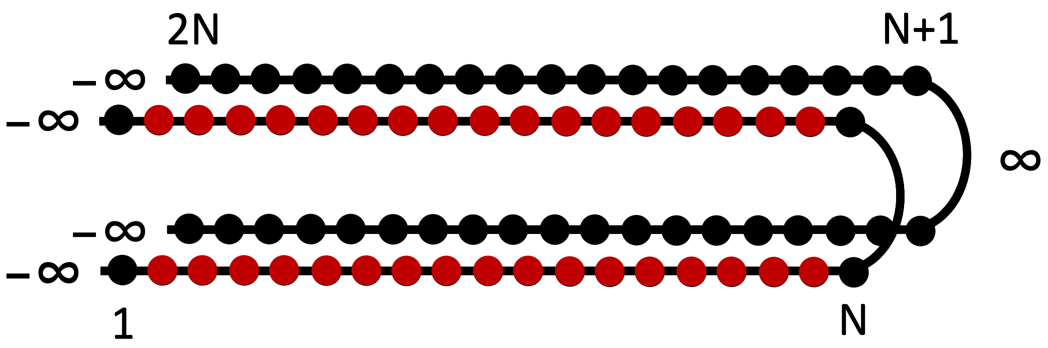




Integrate Out Bath Modes

- Standard Gaussian integral

$$\int \mathcal{D}[\boldsymbol{b}^*, \boldsymbol{b}] = e^{-\boldsymbol{b}^* A \boldsymbol{b} + \boldsymbol{a}^* \cdot \boldsymbol{b} + \boldsymbol{b}^* \cdot \boldsymbol{a}'} = \pi^D \det(A^{-1}) e^{\boldsymbol{a}^* A^{-1} \boldsymbol{a}'}$$



- With

$$S_B[b^*, b] = i b_{2N}^* b_{2N} + i b_1^* b_1 - i\rho(\Omega) b_1^* b_{2N} + i\Delta b_2^* b_1 + i\Delta^* b_{2N}^* b_{2N-1}$$

$$+ i \begin{pmatrix} b_{2N-1} \\ b_{2N-2} \\ b_{2N-3} \\ \vdots \\ b_{N+3} \\ b_{N+2} \\ b_{N+1} \\ \hline b_N \\ b_{N-1} \\ b_{N-2} \\ \vdots \\ b_4 \\ b_3 \\ b_2 \end{pmatrix}^* \left(\begin{array}{ccccccc|ccccccc} 1 & \Delta^* & 0 & \dots & 0 & 0 & 0 & & & & & & & \\ 0 & 1 & \Delta^* & \dots & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & & & & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & & & & 0 & & \\ 0 & 0 & 0 & \dots & 1 & \Delta^* & 0 & & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 & \Delta^* & 0 & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & -1 & 0 & & & & & \\ \hline & & & & & & & 1 & \Delta & 0 & \dots & 0 & 0 & 0 \\ & & & & & & & 0 & 1 & \Delta & \dots & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & & & 0 & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & & & & & & & 0 & 0 & 0 & \dots & 1 & \Delta & 0 \\ & & & & & & & 0 & 0 & 0 & \dots & 0 & 1 & \Delta \\ & & & & & & & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} b_{2N-1} \\ b_{2N-2} \\ b_{2N-3} \\ \vdots \\ b_{N+3} \\ b_{N+2} \\ b_{N+1} \\ \hline b_N \\ b_{N-1} \\ b_{N-2} \\ \vdots \\ b_4 \\ b_3 \\ b_2 \end{pmatrix} \quad (3)$$

$$S_{AB} = |\delta t| \gamma_\Omega \begin{pmatrix} a_{2N} \\ a_{2N-1} \\ a_{2N-2} \\ \vdots \\ a_{N+4} \\ a_{N+3} \\ a_{N+2} \\ \hline 0a_{N+1} \\ -a_N \\ -a_{N-1} \\ \vdots \\ -a_5 \\ -a_4 \\ -a_3 \end{pmatrix}^* \cdot \begin{pmatrix} b_{2N-1} \\ b_{2N-2} \\ b_{2N-3} \\ \vdots \\ b_{N+3} \\ b_{N+2} \\ b_{N+1} \\ \hline b_N \\ b_{N-1} \\ b_{N-2} \\ \vdots \\ b_4 \\ b_3 \\ b_2 \end{pmatrix} + |\delta t| \gamma_\Omega^* \begin{pmatrix} b_{2N-1} \\ b_{2N-2} \\ b_{2N-3} \\ \vdots \\ b_{N+3} \\ b_{N+2} \\ b_{N+1} \\ \hline b_N \\ b_{N-1} \\ b_{N-2} \\ \vdots \\ b_4 \\ b_3 \\ b_2 \end{pmatrix}^* \cdot \begin{pmatrix} a_{2N-2} \\ a_{2N-3} \\ a_{2N-4} \\ \vdots \\ a_{N+2} \\ a_{N+1} \\ 0a_N \\ \hline -a_{N-1} \\ -a_{N-2} \\ -a_{N-3} \\ \vdots \\ -a_3 \\ -a_2 \\ -a_1 \end{pmatrix}$$

Continuum Limit

- Take output of Gaussian integral
- Express as sum, approximate $(1\text{-}idt\ \Omega)^n = e^{i\ n\ dt\ \Omega}$, use δ-function identities
- Convert sum to integral, take dt to 0, replace operators with fields $\hat{b}\text{->}b,\bar{b}$

$$iS_{\text{out}}[a, b_{\text{in}}, b_{\text{out}}, \psi] = iS_A + iS_B + iS_{AB} + iS_\psi \tag{1}$$

$$iS_A = i\int_{t_1}^{t_N} dt\ \left[\bar{a}^+(t)\left(i\partial_t + i\frac{\Gamma}{2}\right)a^+(t) - \bar{a}^-(t)\left(i\partial_t - i\frac{\Gamma}{2}\right)a^-(t) - H_A(\bar{a}^+, a^+) + H_A(\bar{a}^-, a^-)\right] \tag{2}$$

$$iS_B = -[\int_{t_1}^{t_N} dt\ \begin{pmatrix}\bar{b}_{\text{in}}^+(t)\\ \bar{b}_{\text{out}}^+(t)\\ \bar{b}_{\text{out}}^-(t)\\ \bar{b}_{\text{in}}^-(t)\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}b_{\text{in}}^+(t)\\ b_{\text{out}}^+(t)\\ b_{\text{out}}^-(t)\\ b_{\text{in}}^-(t)\end{pmatrix}] + \ln(\rho_B(\bar{b}_{\text{in}}, b_{\text{in}})) \tag{3}$$

$$iS_{AB} = -\sqrt{\Gamma}\int_{t_1}^{t_N} dt\ \bar{a}^+(t)b_{\text{in}}^+(t) - \bar{b}_{\text{out}}^+(t)a^+(t) - \bar{a}^-(t)b_{\text{out}}^-(t) + \bar{b}_{\text{in}}^-(t)a^-(t) \tag{4}$$

Calculating Responses, Overview

- We want to generate moments of b_{out} so we add a source field $iS_\psi = -i\int_{t_1}^{t_N} dt\ \bar{\psi}(t)b^+_{\text{out}}(t) + \bar{b}^-_{\text{out}}(t)\psi(t)$
- We seed the system with an initial state
- Now we integrate out everything except source fields to get the moment-generating function
  - Get Green's functions along the way
- Why Keldysh-Larkin-Ovchinnikov rotate?
  - Go from the contour-ordered Green's functions $G^T, G^{\tilde{T}}, G^<, G^>$ to the causal/anti-causal and Keldysh Green's functions G^R, G^A, G^K

$$S[a, b_{\text{in}}, b_{\text{out}}]$$

$$\downarrow \text{add source field } \psi b_{\text{out}}$$

$$S[a, b_{\text{in}}, b_{\text{out}}, \psi]$$

$$\downarrow \text{integrate out } b_{\text{out}}$$

$$S[a, b_{\text{in}}, \psi]$$

$$\downarrow \text{put in input state}$$

$$S[a, b_{\text{in}}, \psi]$$

$$\downarrow \text{integrate out } b_{\text{in}}$$

$$S[a, \psi]$$

$$\downarrow \text{Larkin rotate}$$

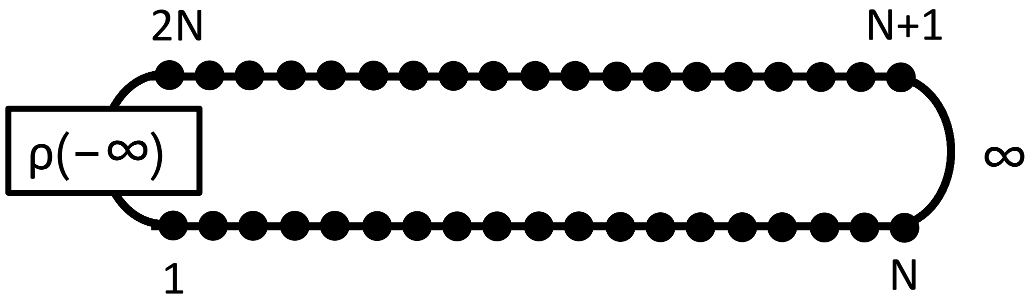
$$S[a, \psi] \rightarrow G^R,\ G^A,\ G^K$$

$$\downarrow \text{integrate out } a$$

$$S[\psi] \rightarrow \text{ calculate moments}$$

Input: T=0 Coherent State

- The input coherent state acts only at the 1st and 2Nth time slice (the density matrix couples these



$$\rho_B(b_{\text{in}}) = \exp\left(\int_{t_1}^{t_N} dt\ f(t)\bar{b}_{\text{in}}^{+} + \bar{f}(t)b_{\text{in}}^{-} - \bar{f}(t)f(t)\right)$$

- How to get this form?

$$\rho_B = \bigotimes_{\Omega} e^{-|\alpha_\Omega|^2}|\alpha_\Omega\rangle\langle\alpha_\Omega|$$

$$\langle\bar{b}_{\text{in}}^{+}|\rho_B|b_{\text{in}}^{-}\rangle = \bigotimes_{\Omega} e^{-|\alpha_\Omega|^2}\langle\bar{b}_{\text{in}}^{+}(\Omega)|\alpha_\Omega\rangle\langle\alpha_\Omega|b_{\text{in}}^{-}(\Omega)\rangle$$

$$= \prod_{k} e^{-|\alpha_\Omega|^2} e^{\bar{b}_{\text{in}}^{+}(\Omega)\alpha_\Omega} e^{\bar{\alpha}_\Omega b_{\text{in}}^{-}(\Omega)}$$

$$= \exp\Bigg(\int_{t_1}^{t_N} dt\Bigg(\bar{b}_{\text{in}}^{+}(t)\overbrace{\int_{\Omega} e^{-i\Omega(t-t_1)}\alpha_\Omega}^{f}$$

$$+ b_{\text{in}}^{-}(t)\underbrace{\int_{\Omega} e^{+i\Omega(t-t_1)}\bar{\alpha}_\Omega}_{\bar{f}} - \int_{\Omega}|\alpha_\Omega|^2\Bigg)\Bigg)$$

- Start with coherent state
- Take matrix element
- Transfer Ω dependence from b to $e^{i\Omega t}$
- Relabel

Green’s Functions

- Assume a system Hamiltonian with one mode $H_A = \Omega_A\, a^{+}a$
- After integrating out all b fields we obtain

$$iS_A = i\int_{t_1}^{t_N} dt \begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \begin{pmatrix} i\Gamma/2 & i\partial_t - \Omega_A \\ i\partial_t - \Omega_A & i\Gamma/2 \end{pmatrix} \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix} \tag{32}$$

$$iS^{\text{left-over}} = i\int_{t_1}^{t_N} dt \; -i\frac{\Gamma}{2} \begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix} - i \begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \cdot \begin{pmatrix} -i\sqrt{\Gamma}\psi\frac{1}{\sqrt{2}} \\ -i\sqrt{\Gamma}\psi\frac{1}{\sqrt{2}} \end{pmatrix} - i \begin{pmatrix} -i\sqrt{\Gamma}\bar{\psi}\frac{1}{\sqrt{2}} \\ -i\sqrt{\Gamma}\bar{\psi}\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix} \tag{33}$$

$$iS_{\text{eff}} = i\int_{t_1}^{t_N} dt \; -\bar{\psi} f + \begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ i\sqrt{\Gamma}\sqrt{2}f \end{pmatrix} - \bar{f}\psi + \begin{pmatrix} 0 \\ -i\sqrt{\Gamma}\sqrt{2}\bar{f} \end{pmatrix} \cdot \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix} \tag{34}$$

- Summing we see the quite nice structure of our cavity Greens functions

$$\begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \begin{pmatrix} 0 & (G^A)^{-1} \\ (G^R)^{-1} & (G^K)^{-1} \end{pmatrix} \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix} = \begin{pmatrix} \bar{a}^{\mathrm{cl}} \\ \bar{a}^{\mathrm{q}} \end{pmatrix} \begin{pmatrix} 0 & i\partial_t - \Omega_A - i\Gamma/2 \\ i\partial_t - \Omega_A - i\Gamma/2 & i\Gamma \end{pmatrix} \begin{pmatrix} a^{\mathrm{cl}} \\ a^{\mathrm{q}} \end{pmatrix}$$

We seem to have switched the definition of a^{cl} and a^{q} somewhere along the derivation

Expectation Values

- $g^{(1)}$ functional $g^{(1)}(t,t') = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \rangle$

$$g^{(1)} = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t') \rangle = i^2 \frac{\delta^2 Z_{\text{out}}}{\delta\bar{\psi}(t)\delta\psi(t')}$$

$$= \int d\tau d\tau' \; (-i\Gamma G^R(t'-\tau')f(\tau') + f(\tau'))(i\Gamma G^A(t-\tau)\bar{f}(\tau) + \bar{f}(\tau))$$

- $g^{(2)}$ functional $g^{(2)}(t,t') = \dfrac{\langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle}{g^{(1)}(t)}$

$$g^{(1)} g^{(2)} = \langle \bar{b}_{\text{out}}(t) b_{\text{out}}(t) \bar{b}_{\text{out}}(t') b_{\text{out}}(t') \rangle = i^4 \frac{\delta^4 Z_{\text{out}}}{\delta\bar{\psi}(t)\delta\psi(t)\delta\bar{\psi}(t')\delta\psi(t')}$$

$$= \ldots$$