Symmetries of Lindbladians

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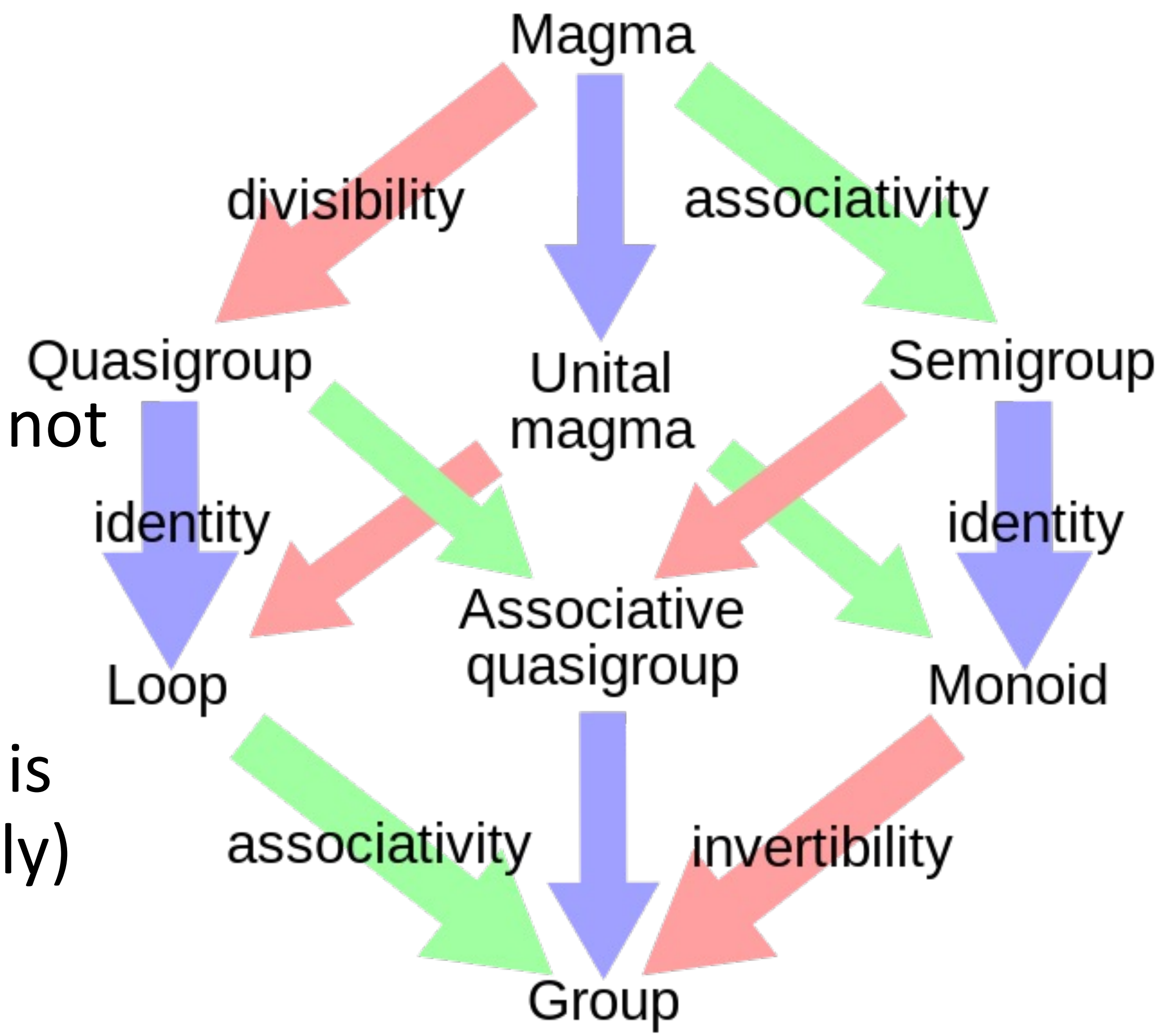
UPenn Condensed Matter Journal Club – 9 October 2023

Here we follow PRA 89, 022118 (2014)

Types of operators

- Linear operators
- Semigroup (such as the CPTP semigroup-–Lindblad): inverses not needed—time evolution can be irreversible
- Unitary group (such as CPTP): inverses exist so time evolution is always reversible (even if unlikely)

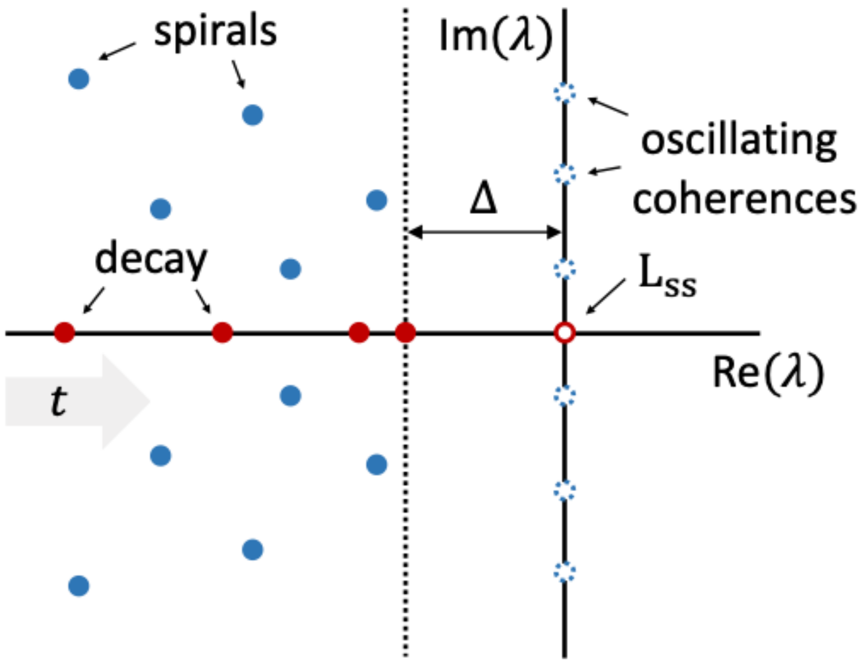


Lindbladian Spectra

- Time evolution by

$$\dot{\rho} = \mathcal{L}(\rho) = -i\,[H,\rho] + \sum_{l=1}^{N^2-1} 2F_l\rho F_l^\dagger - F_l^\dagger F_l\rho - \rho F_l^\dagger F_l$$

- λ are the eigenvalues of *L*
- Δ is the *slowest* dissipative mode: the "dissipative spectral gap"
- Real axis: pure oscillation
- Imag axis: pure decay
- In between: get both
- Note the flipped notion of Re and Im



Results on the steady state

- Eigenvectors of *L* whose eigenvalue is zero (but only some—not closed under addition); Thm: if F has no zero eigvals, unique ss

$$e^{\mathcal{L}t} : \rho_{\mathsf{in}} \xrightarrow{t\to\infty} \rho_{\mathsf{ss}}.$$

- For unitary operators there are N steady states
  - Non-oscillating coherence : sum of projections onto pure states $|\psi_n\rangle\langle\psi_n|$
  - Oscillating coherence (degeneracies): $|\psi_n\rangle\langle\psi_m|$ are ok too
- For finite Lindblad operators there are from 1 to N^2 steady states
  - These can be non-oscillating or oscillating too; work in the rotating frame of H and then there are no oscillations

Some types of steady states

- Examples:

$$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \frac{1}{3} & \cdot & \cdot \\ \cdot & \frac{1}{3} & \cdot \\ \cdot & \cdot & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \rho_{00} & \cdot & \cdot \\ \cdot & \rho_{11} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \rho_{00} & \rho_{01} & \cdot \\ \rho_{10} & \rho_{11} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \rho_{00} & \rho_{01} & \cdot \\ \rho_{10} & \rho_{11} & \cdot \\ \cdot & \cdot & \rho_{22} \end{pmatrix}$$

1. Single pure state steady state
2. Single mixed state steady state
3. Two classical bits
4. One quantum bit (note rho_01=rho_10*)
5. One quantum bit and one classical bit

Conserved Quantities

- Unitary systems
  - $J=J^{T*}$ and $[J,H]=0$
  - So $dJ/dT = 0$
  - Also, $[e^{ixJ}]^{T*}\ H\ e^{ixJ} = 0$

$$\begin{array}{ccc} & [J,H]=0 & \\ \swarrow\!\!\nearrow & & \searrow\!\!\nwarrow \\ \dot{J}=0 & \Leftrightarrow & U^{\dagger}HU=H \end{array}$$

- Lindblad systems
  - Impose $[J,H]=[J,F]=0$
    - **Satisfied by J such that $\boldsymbol{L}^{T*}J=0$**
  - Then get conserved quantities
  - and conserved Lindbladian

$$\begin{array}{ccc} & [J,H]=[J,F_l]=0 \quad \forall l & \\ \swarrow & & \searrow \\ \dot{J}=\mathcal{L}^{\dagger}(J)=0 & & \mathcal{U}^{\dagger}\mathcal{L}\mathcal{U}=\mathcal{L} \end{array}$$

- Symmetry != conserved quantity!

Subspace symmetries

- In a D dimensional steady state subspace there are D conserved quantities J such that $\rho_\mu = \mathrm{Tr}\{J_\mu^\dagger \rho_{\mathsf{in}}\}.$
  - “the number of conserved quantities is equal to the dimension of the Lie algebra of subspace symmetries”
- These generate U=e^{ixJ} where $e^{\mathcal{L}t}(U^\dagger \rho_{\mathsf{in}} U) = U^\dagger e^{\mathcal{L}t}(\rho_{\mathsf{in}})U,$ U and time evolution commute
- Continuous symmetries: Unitary rotations in subspaces
- Discrete symmetries: exchange of subspaces

Global symmetries

- Symmetries of both L and L^{T*} which satisfy $\mathcal{U}^\dagger \mathcal{L} \mathcal{U} = \mathcal{L}$,
- Example:
  - Permute jump operators
  - $[H,U]=0$ and $UFU = e^{if}F$

Anti-Unitary Symmetries

- TRS and Chiral symmetry– see New J. Phys 15, 085001 (2013)
- Contour-reversal symmetry $\mathcal{T}^{-1}[iL(\boldsymbol{k})]^*\mathcal{T} = iL(-\boldsymbol{k}).$
  - ST and MC, PRB 106, L161109 (2022)

$$L_{\text{dis}} = \frac{\Gamma}{2}\begin{pmatrix} A_{\boldsymbol{k}} - B_{\boldsymbol{k}} & -2B_{\boldsymbol{k}} & C_{\boldsymbol{k}} - C_{-\boldsymbol{k}}^{\top} & 2C_{-\boldsymbol{k}}^{\top} \\ -2A_{\boldsymbol{k}} & B_{\boldsymbol{k}} - A_{\boldsymbol{k}} & -2C_{\boldsymbol{k}} & C_{\boldsymbol{k}} - C_{-\boldsymbol{k}}^{\top} \\ C_{\boldsymbol{k}}^{\dagger} - C_{-\boldsymbol{k}}^{*} & -2C_{-\boldsymbol{k}}^{*} & B_{-\boldsymbol{k}}^{\top} - A_{-\boldsymbol{k}}^{\top} & 2A_{-\boldsymbol{k}}^{\top} \\ 2C_{\boldsymbol{k}}^{\dagger} & C_{\boldsymbol{k}}^{\dagger} - C_{-\boldsymbol{k}}^{*} & 2B_{-\boldsymbol{k}}^{\top} & A_{-\boldsymbol{k}}^{\top} - B_{-\boldsymbol{k}}^{\top} \end{pmatrix}$$

Thoughts on my mind

- It seems like there are different ways to approach conserved quantities, why approach it as Albert and Jiang did?
- Is there something we can say about the response of a steady state with symmetries?

Symmetries and TIs

- Chiral $SHS^{-1}=-H$
  - for unitary S
- Time-reversal $THT^{-1}=H$
  - for anti-unitary T
- Charge-conjugation $CHC^{-1}=-H$
  - for antiunitary C
- Non-Hermitian
  - $SHS^{T*} = -H$
  - $CH^{*}C^{T*} = -H$
  - $TH^{*}T^{T*}= H$

Symmetry				Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Operator evolution

- Schrodinger picture (trace preserving)

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}\left\{ L_i^\dagger L_i, \rho \right\} \right)$$

- Heisenberg picture (identity preserving)

$$\dot{X} = \frac{i}{\hbar}[H,X] + \sum_i \gamma_i \left(L_i^\dagger X L_i - \frac{1}{2}\left\{ L_i^\dagger L_i, X \right\} \right)$$