

Hamiltonian

We are given the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 + \frac{\alpha}{4}x^4 - F \cos(\omega t)x \quad (1)$$

with the quantization condition

$$x = \sqrt{\hbar/2} \frac{1}{\sqrt{m\omega_0}}(a^\dagger + a), \quad p = i\sqrt{\hbar/2}\sqrt{m\omega_0}(a^\dagger - a) \quad (2)$$

we have

$$\frac{p^2}{2m} = \frac{1}{4}\hbar\omega_0(-a^\dagger a^\dagger + aa^\dagger + a^\dagger a - aa) \quad (3)$$

$$\frac{m\omega_0^2}{2}x^2 = \frac{1}{4}\hbar\omega_0(a^\dagger a^\dagger + aa^\dagger + a^\dagger a + aa) \quad (4)$$

and the canonical commutation $[a, a^\dagger] = \mathbf{1}$, $[a^\dagger, a^\dagger] = 0$, and $[a, a] = 0$ which means

$$aa^\dagger = \mathbf{1} + a^\dagger a \quad (5)$$

using these we find

$$\frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 = \hbar\omega_0(a^\dagger a + \frac{1}{2}) \quad (6)$$

Now,

$$\frac{\alpha}{4}x^4 = \frac{\alpha}{4} \frac{\hbar^2}{4m^2\omega_0^2} (a^\dagger + a)(a^\dagger + a)(a^\dagger + a)(a^\dagger + a) \quad (7)$$

$$= \frac{\alpha}{4} \frac{\hbar^2}{4m^2\omega_0^2} (a^\dagger a^\dagger a^\dagger a^\dagger + 4a^\dagger a^\dagger a^\dagger a + 6a^\dagger a^\dagger aa + 4a^\dagger aaa + aaaa + 6(a^\dagger a^\dagger + 2a^\dagger a + aa) + 3) \quad (8)$$

where in the second line we normal ordered and collected terms. Now this is rather more terms than we would like to deal with and we notice the appearance of superconducting-like terms aa and $a^\dagger a^\dagger$. Let us ignore all of the terms except for the one highlighted in blue, which is the term that we will use for a non-linearity in the quantum system. We now have the quantized Hamiltonian

$$H = \hbar\omega_0 a^\dagger a + K a^\dagger a^\dagger aa + A \cos(\omega t)(a^\dagger + a) \quad (9)$$

for appropriate constants K and A

Putting it on the Keldysh contour

Now we can put this on the Keldysh contour using the generic relation (see Eq. 27-28 Sieberer et al. [2016])

$$iS = i \int_{-\infty}^{\infty} dt \left(\bar{a}^+ i \partial_t \mathbf{1} a^+ - \bar{a}^- i \partial_t \mathbf{1} a^- + \mathcal{L} \right) \quad (10)$$

for some appropriate representation of the Louivillian \mathcal{L} . We note that \bar{a} and a are now scalars since in putting on the Keldysh contour we have inserted complete sets of coherent states and \bar{a} and a are the eigenvalues of the operators a^\dagger and a respectively. The superscript \pm refer to the right-moving and left-moving contours respectively. Now what should \mathcal{L} be? Let us consider

$$\mathcal{L} = \mathcal{L}_H + \mathcal{L}_1 + \mathcal{L}_2 \quad (11)$$

describing the Hamiltonian, 1-photon loss, and 2-photon loss terms. We have

$$\mathcal{L}_H = -H(\bar{a}^+, a^+) + H(\bar{a}^-, a^-) \quad (12)$$

where

$$H(\bar{a}^+, a^+) = \hbar\omega_0 \bar{a}^+ a^+ + K \bar{a}^+ \bar{a}^+ a^+ a^+ + A \cos(\omega t) (\bar{a}^+ + a^+) \quad (13)$$

and the dissipators are

$$\mathcal{L}_1 = \frac{i}{2} \gamma_1 \left(\bar{a}^+ a^+ + \bar{a}^- a^- - 2a^+ \bar{a}^- \right) \quad (14)$$

and

$$\mathcal{L}_2 = \frac{i}{2} \gamma_2 \left(\bar{a}^+ \bar{a}^+ a^+ a^+ + \bar{a}^- \bar{a}^- a^- a^- - 2a^+ a^+ \bar{a}^- \bar{a}^- \right) \quad (15)$$

where we have assumed the form of Lindblad dissipators.

Keldysh rotation

Now we introduce the Keldysh rotation for an arbitrary field ϕ

$$\phi^c = \frac{1}{\sqrt{2}} (\phi^+ + \phi^-), \quad \phi^q = \frac{1}{\sqrt{2}} (\phi^+ - \phi^-) \quad (16)$$

where for the bar fields the relation is the same (but not so in the case of fermions where one needs to Larkin-Ovchinnikov rotate). From this we can see that

$$\phi^+ = \frac{1}{\sqrt{2}} (\phi^c + \phi^q), \quad \phi^- = \frac{1}{\sqrt{2}} (\phi^c - \phi^q) \quad (17)$$

From which in short order (aided by Mathematica, where we don't need to worry about commutation relations since for Bosons the scalars are complex numbers which commute—for fermions the

scalars are Grassman which do not commute)

$$\bar{a}^+ + a^+ - \bar{a}^- - a^- = \sqrt{2}(\bar{a}^q + a^q) \quad (18)$$

$$\bar{a}^+ a^+ - \bar{a}^- a^- = \bar{a}^c a^q + \bar{a}^q a^c \quad (19)$$

$$\bar{a}^+ a^+ + \bar{a}^- a^- - 2a^+ \bar{a}^- = \bar{a}^q a^c - \bar{a}^c a^q + 2\bar{a}^q a^q \quad (20)$$

$$\bar{a}^+ \bar{a}^+ a^+ a^+ - \bar{a}^- \bar{a}^- a^- a^- = +\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q + \bar{a}^q \bar{a}^q a^c a^q \quad (21)$$

$$\bar{a}^+ \bar{a}^+ a^+ a^+ + \bar{a}^- \bar{a}^- a^- a^- - 2a^+ a^+ \bar{a}^- \bar{a}^- = -\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q - \bar{a}^q \bar{a}^q a^c a^q + 4\bar{a}^c \bar{a}^q a^c a^q \quad (22)$$

So we are now in a position to write the action in terms of the classical-quantum field combinations (c.f. Kamenev Eq. 2.56)

$$\begin{aligned} iS = i \int_{-\infty}^{\infty} dt \left((i\partial_t - \hbar\omega_0)(\bar{a}^c a^q + \bar{a}^q a^c) - \sqrt{2}A \cos(\omega t)(\bar{a}^q + a^q) + \frac{i}{2}\gamma_1(\bar{a}^q a^c - \bar{a}^c a^q + 2\bar{a}^q a^q) \right. \\ \left. - K(\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q + \bar{a}^q \bar{a}^q a^c a^q) \right. \\ \left. + \frac{i}{2}\gamma_2(-\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q - \bar{a}^q \bar{a}^q a^c a^q + 4\bar{a}^c \bar{a}^q a^c a^q) \right) \quad (23) \end{aligned}$$

Now from here we could follow the approach of Eq. 29-33 of [Soriente et al. \[2021\]](#) to find a mean-field solution from an ansatz ($a_c = \alpha$ and $a_q = 0$) and the condition that $\delta iS/\delta \bar{a}^q = 0$ (where $a_q \rightarrow 0$ after the functional derivative). Instead we shall take the in Eq. 107 of [McDonald and Clerk \[2023\]](#) which first repackages in terms of $n = \bar{a}^c a^c + \bar{a}^q a^q = \bar{a}^- a^- + \bar{a}^+ a^+$ and then identifies n with a thermal Bose distribution. We note that

$$\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q + \bar{a}^q \bar{a}^q a^c a^q = (\bar{a}^c a^c + \bar{a}^q a^q)(\bar{a}^c a^q + \bar{a}^q a^c) = n(\bar{a}^c a^q + \bar{a}^q a^c) \quad (24)$$

and by a similar deduction we identify $n' = \bar{a}^c a^c - \bar{a}^q a^q$

$$-\bar{a}^c \bar{a}^c a^c a^q + \bar{a}^c \bar{a}^q a^c a^c + \bar{a}^q \bar{a}^c a^q a^q - \bar{a}^q \bar{a}^q a^c a^q + 4\bar{a}^c \bar{a}^q a^c a^q = (\bar{a}^q a^c - \bar{a}^c a^q)(\bar{a}^c a^c - \bar{a}^q a^q) + 4\bar{a}^c \bar{a}^q a^c a^q \quad (25)$$

$$= (n' + 2\bar{a}^c a^q)\bar{a}^q a^c - (n' - 2\bar{a}^q a^c)\bar{a}^c a^q \quad (26)$$

where back in contour space

$$n' + 2\bar{a}^c a^q = 2\bar{a}^- a^+ - \bar{a}^- a^- + \bar{a}^+ a^+ = 2\bar{a}^+ a^+ - L_1 = L^+ \quad (27)$$

$$n' - 2\bar{a}^q a^c = 2\bar{a}^- a^+ + \bar{a}^- a^- - \bar{a}^+ a^+ = 2\bar{a}^- a^- - L_1 = L^- \quad (28)$$

where we set $L_1 = \bar{a}^+ a^+ + \bar{a}^- a^- = 2a^+ \bar{a}^-$ and defined L^\pm accordingly. With these definitions and the warning that the mean field solution may not be valid we can continue to rewrite

$$\begin{aligned} iS = i \int_{-\infty}^{\infty} dt \left((i\partial_t - \hbar\omega_0 - A \cos(\omega t))(\bar{a}^c a^q + \bar{a}^q a^c) + \frac{i}{2}\gamma_1(\bar{a}^q a^c - \bar{a}^c a^q + 2\bar{a}^q a^q) \right. \\ \left. - nK(\bar{a}^c a^q + \bar{a}^q a^c) \right. \\ \left. + \frac{i}{2}\gamma_2(L^+ \bar{a}^q a^c - L^- \bar{a}^c a^q) \right) \quad (29) \end{aligned}$$

Now we modified the time driving so that it is also quadratic $\sqrt{2}A \cos(\omega t)(\bar{a}^q + a^q) \rightarrow A \cos(\omega t)(\bar{a}^c a^q + \bar{a}^q a^q) = A \cos(\omega t)(\bar{a}^+ a^+ - \bar{a}^- a^-)$ which comes from a term like $A \cos(\omega t)x^2$ rather than one like $A \cos(\omega t)x$. Repackaging this as a quadratic form

$$iS = i \int_{-\infty}^{\infty} dt \begin{pmatrix} \bar{a}_c \\ \bar{a}_q \end{pmatrix} \begin{pmatrix} 0 & i\partial_t - \hbar\omega_0 - A \cos(\omega t) - nK - i\gamma_1/2 - i\gamma_2 L^-/2 \\ i\partial_t - \hbar\omega_0 - A \cos(\omega t) - nK + i\gamma_1/2 + i\gamma_2 L^+/2 & i\gamma_1 \end{pmatrix} \begin{pmatrix} a_c \\ a_q \end{pmatrix} \quad (30)$$

where we had no need to write in Bogoliubov-de Gennes Nambu form since here we don't have superconducting terms. Now we Fourier transform to frequency space $i\partial_t \mapsto \Omega$ and $\cos(\omega t) \mapsto [\delta(\Omega - \omega) + \delta(\Omega + \omega)]/2$

From this we can immediately read off the Green's functions

$$G^R = \left[\Omega - \hbar\omega_0 - \frac{A}{2}[\delta(\Omega - \omega) + \delta(\Omega + \omega)] - nK + i\frac{\gamma_1}{2} + i\frac{\gamma_2}{2}L^+ \right]^{-1} \quad (31)$$

Everything we have done so far is exact and we have not taken the mean field assumption yet. Now let us take it: assume that n is a time-independent scalar valued function (constant)—this is the truncation $n(t) = n_0 + n_1 \cos(\omega t) + \dots$ to the zeroth order term. Additionally since I am uneasy setting L^+/L^- to a constant, let us take $\gamma_2 = 0$ for now. Finally, let us approximate the delta function as a Lorentzian with broadening η (such as frequency instability of a function generator, or the linewidth of emission from a laser) so that $\delta(\Omega - \omega) = \frac{1}{\pi} \frac{\eta}{(\Omega - \omega)^2 + \eta^2}$. We then obtain the retarded Green's function as a simple fraction of scalars

$$G^R = 1 \left/ \left[\Omega - (\hbar\omega_0 + n_0 K) - \frac{A}{2\pi} \left[\frac{\eta}{(\Omega - \omega)^2 + \eta^2} + \frac{\eta}{(\Omega + \omega)^2 + \eta^2} \right] + i\frac{\gamma_1}{2} \right] \right. \quad (32)$$

advanced $G^A = (G^R)^\dagger$ (where L^+ and L^- are related by \dagger which not only does transposition and complex conjugation, but contour exchange) and the Keldysh Green's function

$$G^K = G^R \Sigma^K G^A \quad (33)$$

with Keldysh self-energy $\Sigma^K = -i\gamma_1$.

From these Green's functions we can obtain the spectral function

$$\mathcal{A}(\Omega) = -2 \text{Im}(G^R(\Omega)) \quad (34)$$

density

$$\rho(\Omega) = \langle a^\dagger a \rangle = \frac{1}{2} (iG^K(\Omega) - 1) \quad (35)$$

where we note that $G^R - G^A$ is a constant.

The $g^{(2)}$ functional requires some more careful consideration that these key functions since while the system appears free, it isn't actually, so Wick's theorem doesn't hold.

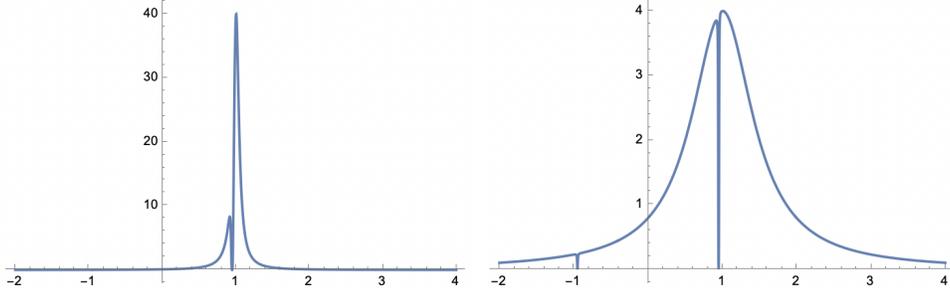


FIG. 1: spectral function for $\gamma_1 = 0.1$ (left), $\gamma_1 = 1$ (right). Other parameters are $\hbar\omega_0 = 1$, $n_0 = \pi$ (guessed self-consistent point), $K = 0$, $A = 0.1$, $\omega = 0.95$, $\eta = 0.001$. x-axis is Ω and y-axis is $\mathcal{A}(\Omega)$.

Note that in this approximation, $n_0 K$ is just a shift to the resonant frequency.

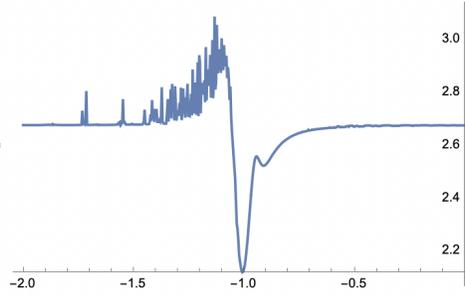


FIG. 2: number function $\int_{\Omega} \rho(\Omega)$ as a function of drive frequency ω for $\gamma_1 = 0.1$, $\hbar\omega_0 = 1$, $n_0 = \pi$, $K = 0$, $A = 0.1$, $\eta = 0.001$. x-axis is Ω and y-axis is $\mathcal{A}(\Omega)$. Note the dip at $-\hbar\omega_0$.

References

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