Keldysh Approach to Input States for Cavity Quantum Materials

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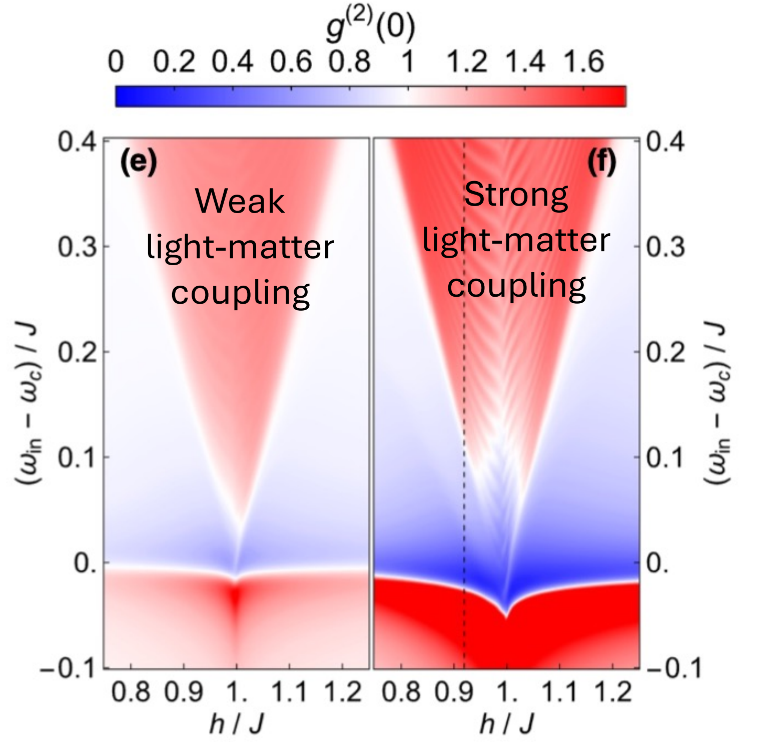
Outline

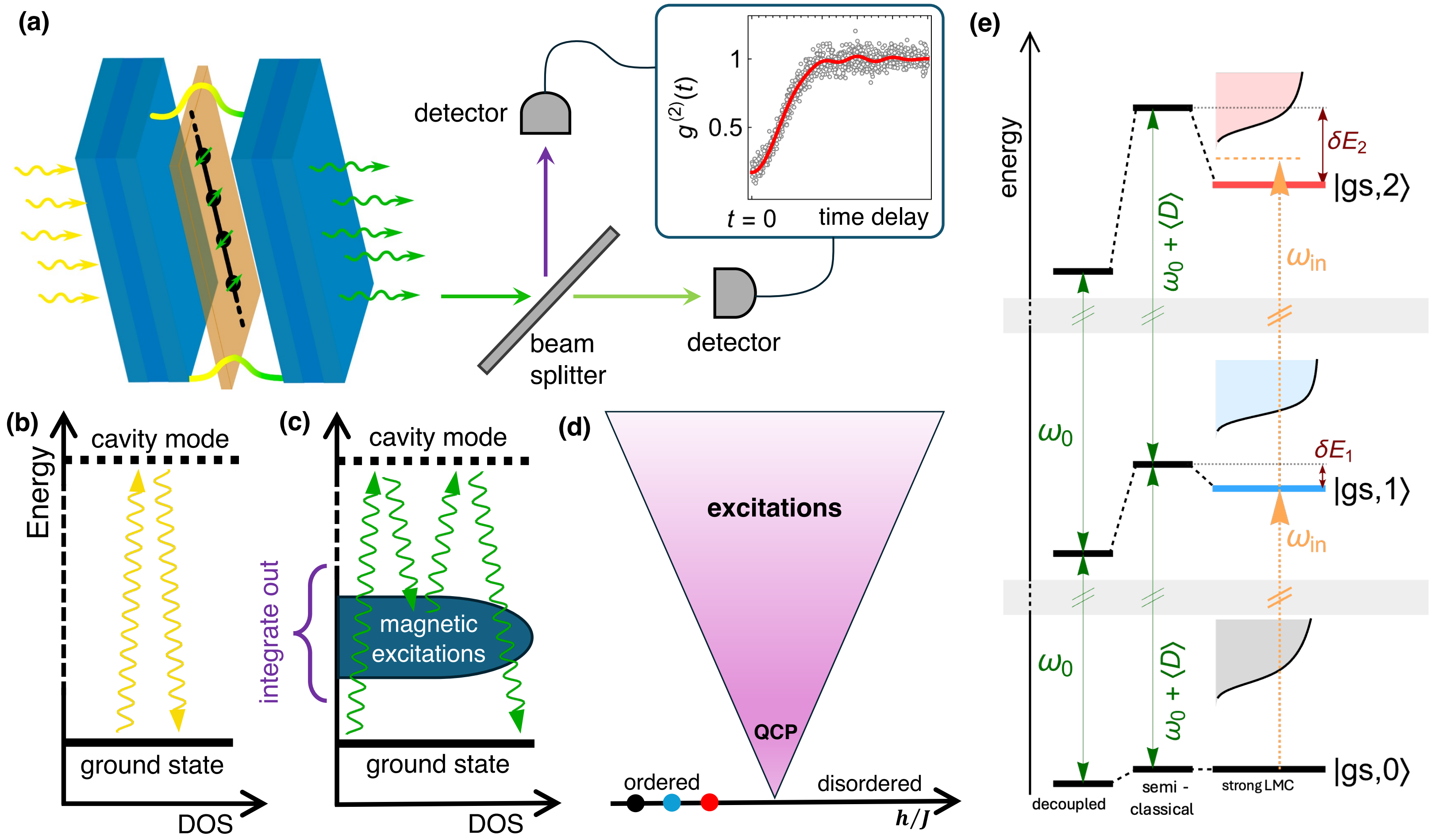
- Background: Input-Output theory
  - Ben’s work: quantum critical materials in cavities
  - Spenser’s work: Higgs polaritons in cavities
  - Input-output Keldysh action
  - Ahana’s work: non-thermal states in Keldysh
- Directions
  - Chiral materials and entangled states of light
  - Spin liquids, (anti)bunching, and squeezed states of light
  - *Other ideas?*

Background 1: Ben's Quantum Criticality Work

- arXiv:2411.08964

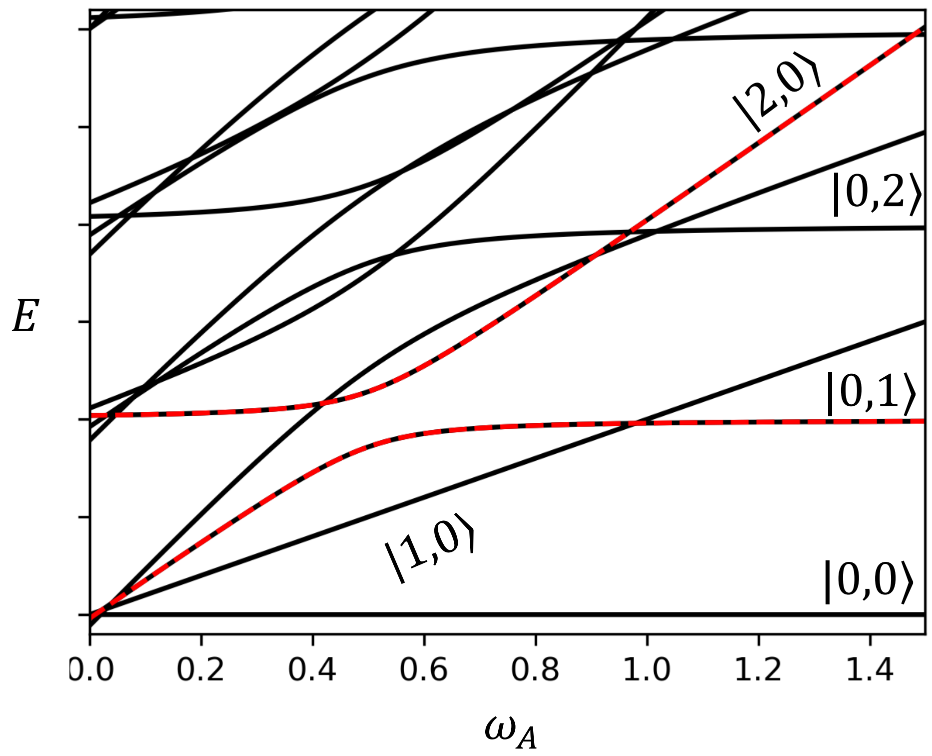
$$g^{(2)}(t) = \frac{\langle : \mathcal{E}^-(\tau)\mathcal{E}^-(\tau+t)\mathcal{E}^+(\tau+t)\mathcal{E}^+(\tau) : \rangle}{\langle : \mathcal{E}^-(\tau)\mathcal{E}^+(\tau) : \rangle \langle : \mathcal{E}^-(\tau+t)\mathcal{E}^+(\tau+t) : \rangle}.$$

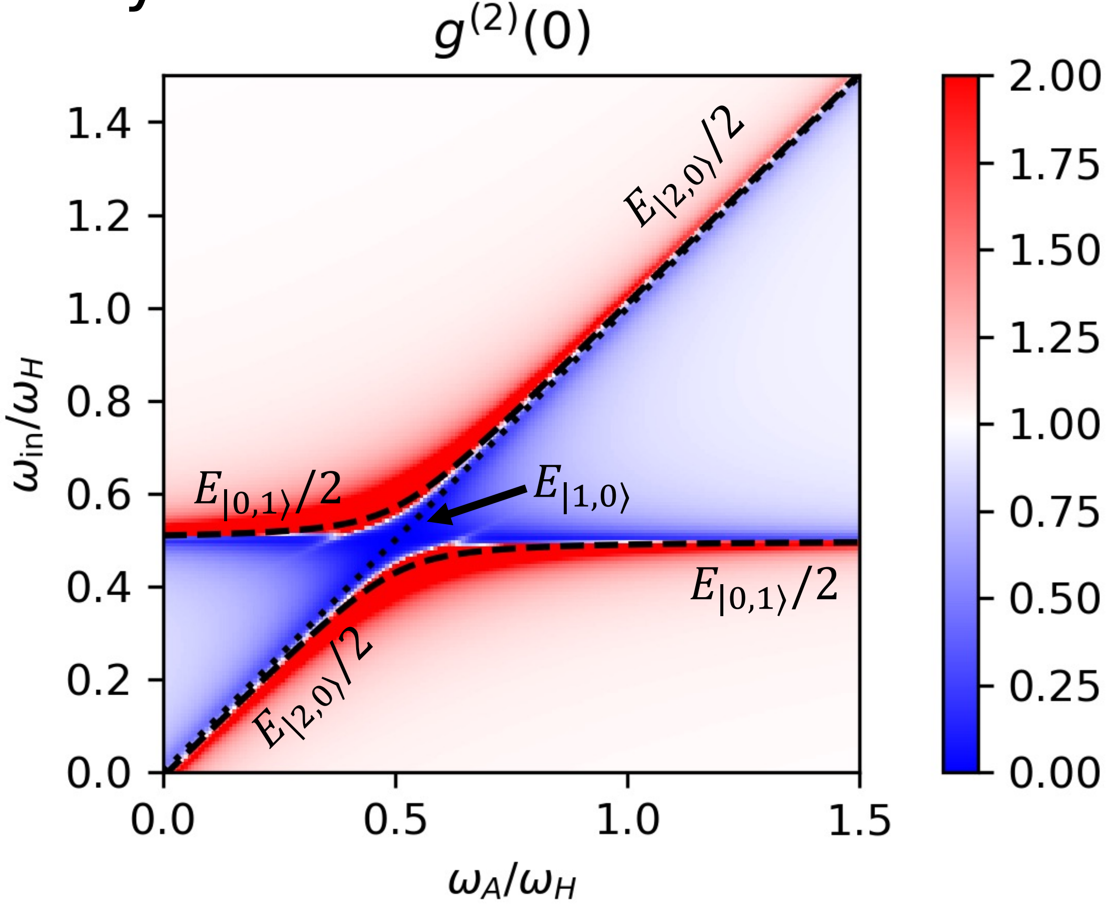




Background 2: Spenser's Higgs Polariton Work

- Forthcoming
- Classical input→ (anti)bunched output
- Matter (Higgs) as an effective nonlinearity
- Method: master equation/P function



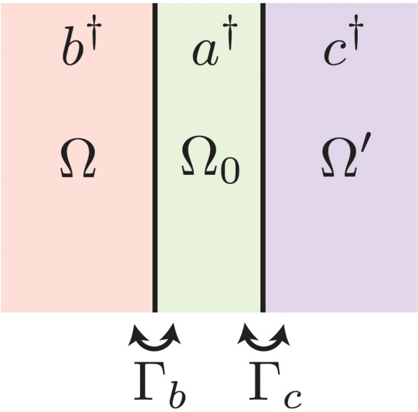


Background 3: Keldysh Input-Output Action

- Partition function is $\mathrm{Tr}[\rho] = 1$

$$Z = \mathrm{Tr}[U(\infty,0)\rho(0)U^\dagger(\infty,0)]$$
$$= \int \mathcal{D}[a^\pm, b^\pm_{\rm in}]\, e^{iS[a^\pm]} \langle \bar{b}^+_{\rm in}|\rho_B(0)|b^-_{\rm in}\rangle$$

- Ex. Initial state $\rho(0) = \rho_B(0) \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|.$
- Action for transmission through a cavity



$$
\begin{aligned}
iS[a, b^\pm_{\rm in}, b^\pm_{\rm out}, c^\pm_{\rm in}, c^\pm_{\rm out}] &= i\int dt \begin{pmatrix}\bar{a}^+ \\ \bar{a}^-\end{pmatrix}\begin{pmatrix}-i\partial_t - H_A + i\Gamma & 0 \\ 0 & i\partial_t + H_A + i\Gamma\end{pmatrix}\begin{pmatrix}a^+ \\ a^-\end{pmatrix} \\
&- \begin{pmatrix}\bar{b}^+_{\rm in} \\ \bar{b}^+_{\rm out} \\ \bar{b}^-_{\rm out} \\ \bar{b}^-_{\rm in}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 \\ i\mathcal{G}_b(t_{\rm out}-t_{\rm in}) & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & i\mathcal{G}_b(t_{\rm in}-t_{\rm out}) & 1\end{pmatrix}\begin{pmatrix}b^+_{\rm in} \\ b^+_{\rm out} \\ b^-_{\rm out} \\ b^-_{\rm in}\end{pmatrix} \\
&+ \int dt \begin{pmatrix}\bar{b}^+_{\rm in} \\ \bar{b}^-_{\rm in}\end{pmatrix}\begin{pmatrix}0 \\ \sqrt{\Gamma}\mathcal{G}_b(t_{\rm in}-t)a^-\end{pmatrix} + \begin{pmatrix}-\sqrt{\Gamma}\bar{a}^+\mathcal{G}_b(t-t_{\rm in}) \\ 0\end{pmatrix}\begin{pmatrix}b^+_{\rm in} \\ b^-_{\rm in}\end{pmatrix} \\
&+ \int dt \begin{pmatrix}\bar{b}^+_{\rm out} \\ \bar{b}^-_{\rm out}\end{pmatrix}\begin{pmatrix}-\sqrt{\Gamma}\mathcal{G}_b(t_{\rm out}-t)a^+ \\ 0\end{pmatrix} + \begin{pmatrix}0 \\ \sqrt{\Gamma}\bar{a}^-\mathcal{G}_b(t-t_{\rm out})\end{pmatrix}\begin{pmatrix}b^+_{\rm out} \\ b^-_{\rm out}\end{pmatrix} \\
&- \begin{pmatrix}\bar{c}^+_{\rm in} \\ \bar{c}^+_{\rm out} \\ \bar{c}^-_{\rm out} \\ \bar{c}^-_{\rm in}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 \\ i\mathcal{G}_c(t_{\rm out}-t_{\rm in}) & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & i\mathcal{G}_c(t_{\rm in}-t_{\rm out}) & 1\end{pmatrix}\begin{pmatrix}c^+_{\rm in} \\ c^+_{\rm out} \\ c^-_{\rm out} \\ c^-_{\rm in}\end{pmatrix} \\
&+ \int dt \begin{pmatrix}\bar{c}^+_{\rm in} \\ \bar{c}^-_{\rm in}\end{pmatrix}\begin{pmatrix}0 \\ \sqrt{\Gamma}\mathcal{G}_c(t_{\rm in}-t)a^-\end{pmatrix} + \begin{pmatrix}-\sqrt{\Gamma}\bar{a}^+\mathcal{G}_c(t-t_{\rm in}) \\ 0\end{pmatrix}\begin{pmatrix}c^+_{\rm in} \\ c^-_{\rm in}\end{pmatrix} \\
&+ \int dt \begin{pmatrix}\bar{c}^+_{\rm out} \\ \bar{c}^-_{\rm out}\end{pmatrix}\begin{pmatrix}-\sqrt{\Gamma}\mathcal{G}_c(t_{\rm out}-t)a^+ \\ 0\end{pmatrix} + \begin{pmatrix}0 \\ \sqrt{\Gamma}\bar{a}^-\mathcal{G}_c(t-t_{\rm out})\end{pmatrix}\begin{pmatrix}c^+_{\rm out} \\ c^-_{\rm out}\end{pmatrix}
\end{aligned}
$$

- Ex: thermal coherent state input

$$
\begin{aligned}
iS[a,f,T] &= i\int dt \begin{pmatrix}\bar{a}^{\rm cl} \\ \bar{a}^{\rm q}\end{pmatrix}\begin{pmatrix}0 & -i\partial_t - H_A - i\Gamma \\ -i\partial_t - H_A + i\Gamma & 2i\Gamma + i\frac{\Gamma}{2}\frac{z}{1-z}\end{pmatrix}\begin{pmatrix}a^{\rm cl} \\ a^{\rm q}\end{pmatrix} \\
&+ \int dt \begin{pmatrix}\bar{a}^{\rm cl} \\ \bar{a}^{\rm q}\end{pmatrix}\begin{pmatrix}-i\sqrt{\Gamma}\psi^{\rm q} \\ i\sqrt{\Gamma}\psi^{\rm q} + \sqrt{\Gamma/2}f\end{pmatrix} + \int dt \begin{pmatrix}-i\sqrt{\Gamma}\bar{\psi}^{\rm q} \\ -i\sqrt{\Gamma}\bar{\psi}^{\rm q} - \bar{f}\sqrt{\Gamma/2}\end{pmatrix}\begin{pmatrix}a^{\rm cl} \\ a^{\rm q}\end{pmatrix}
\end{aligned}
$$

See also: Aaron Daniel Thesis (U. Basel 2023)

Background 4: Ahana's Work

- PRB 99, 054306 (2019): Key formalism development
  - PRL 127, 200603 (2021): Bosonic Renyi entropies
  - PRX-Q 6, 020341 (2025): photon mediated superconductivity

TABLE I. Modification in the structure of the Keldysh field theory to incorporate arbitrary initial density matrix, $\hat{\rho}_0$ for bosonic and fermionic systems: the matrix element of $\hat{\rho}_0$ is added as a quadratic term, $\delta S(u)$ in the action, where a function of the initial source $\hat{u}$ couples to the bilinears of the initial quantum fields $\phi_q^*\phi_q$ for bosons and $\psi_1^*\psi_2$ for fermions. $\mathcal{N}(u)$ is the normalization of the partition function obtained from the modified action $S+\delta S(u)$ and physical correlation functions are obtained by taking the set of derivatives $\mathcal{L}(\partial_u, \rho_0)$, completely dictated by $\hat{\rho}_0$, of $\mathcal{N}(u)\hat{G}^{(n)}(u)$, where $\hat{G}^{(n)}(u)$ is the "$n-$particle Green's function" in presence of the initial source $\hat{u}$. For the generic density matrix of a multimode system, $\hat{\rho}_0 = \sum_{nm} c_{nm}|\{n\}\rangle\langle\{m\}|$ with $N=\sum_\gamma n_\gamma = \sum_\gamma m_\gamma$, $\partial_{\alpha_j\beta_j}$ denotes partial derivative with respect to $u_{\alpha_j\beta_j}$ which couples to the jth pair of the fields with indices (α_j, β_j). In case of fermions, the set $\mathcal{A}$ denotes the set of occupied modes in the initial $\hat{\rho}_0$.

System	Initial density matrix	$\delta S(u)$	$\mathcal{N}(u)$	$\mathcal{L}(\partial_u, \rho_0)$
boson	Single mode: diagonal $\hat{\rho}_0 = \sum_n c_n\|n\rangle\langle n\|$	$\mathbf{i}\phi_q^*(0)\phi_q(0)\frac{1+u}{1-u}$	$\frac{1}{1-u}$	$\sum_n \frac{1}{n!}c_n\partial_u^n$
	Multimode: diagonal $\hat{\rho}_0 = \sum_{\{n\}} c_{\{n\}}\|\{n\}\rangle\langle\{n\}\|$	$\mathbf{i}\sum_\alpha \phi_q^*(\alpha,0)\phi_q(\alpha,0)\frac{1+u_\alpha}{1-u_\alpha}$	$\frac{1}{\prod_\alpha(1-u_\alpha)}$	$\sum_{\{n\}} c_{\{n\}}\prod_\gamma \frac{\partial_{u_\gamma}^{n_\gamma}}{n_\gamma!}$
	Multimode: generic $\hat{\rho}_0 = \sum_{nm} c_{nm}\|\{n\}\rangle\langle\{m\}\|$	$\mathbf{i}\sum_{\alpha\beta}\phi_q^*(\alpha,0)\phi_q(\beta,0)[2(1-\hat{u})^{-1}-1]_{\alpha\beta}$	$\mathrm{Det}(1-\hat{u})^{-1}$	$\sum_{nm} c_{nm}\prod_\alpha \frac{1}{\sqrt{n_\alpha!m_\alpha!}}\prod_j \partial_{\alpha_j\beta_j}$
fermion	Single mode: diagonal $\hat{\rho}_0 = \sum_{n=0,1} c_n\|n\rangle\langle n\|$	$\mathbf{i}\psi_1^*(0)\psi_2(0)\frac{1-u}{1+u}$	$1+u$	$c_0 + c_1\frac{\partial}{\partial u}$
	Multimode: diagonal $\hat{\rho}_0 = \sum_{\{n\}} c_{\{n\}}\|\{n\}\rangle\langle\{n\}\|$	$\mathbf{i}\sum_\alpha \psi_1^*(\alpha,0)\psi_2(\alpha,0)\frac{1-u_\alpha}{1+u_\alpha}$	$\prod_\alpha(1+u_\alpha)$	$\sum_{\{n\}} c_{\{n\}}\prod_{\gamma\in\mathcal{A}}\partial_{u_\gamma}$
	Multimode: generic $\hat{\rho}_0 = \sum_{nm} c_{nm}\|\{n\}\rangle\langle\{m\}\|$	$\mathbf{i}\sum_{\alpha\beta}\psi_1^*(\alpha,0)\psi_2(\beta,0)[2(1+\hat{u})^{-1}-1]_{\alpha\beta}$	$\mathrm{Det}(1+\hat{u})$	$\sum_{nm} c_{nm}\prod\partial_{\alpha_j\beta_j}$

Fock states are complete!

$$\rho = \sum_{\{n,m\}} c_{nm}|n\rangle\langle m|$$

Example: Fock state input in single mode bosonic system

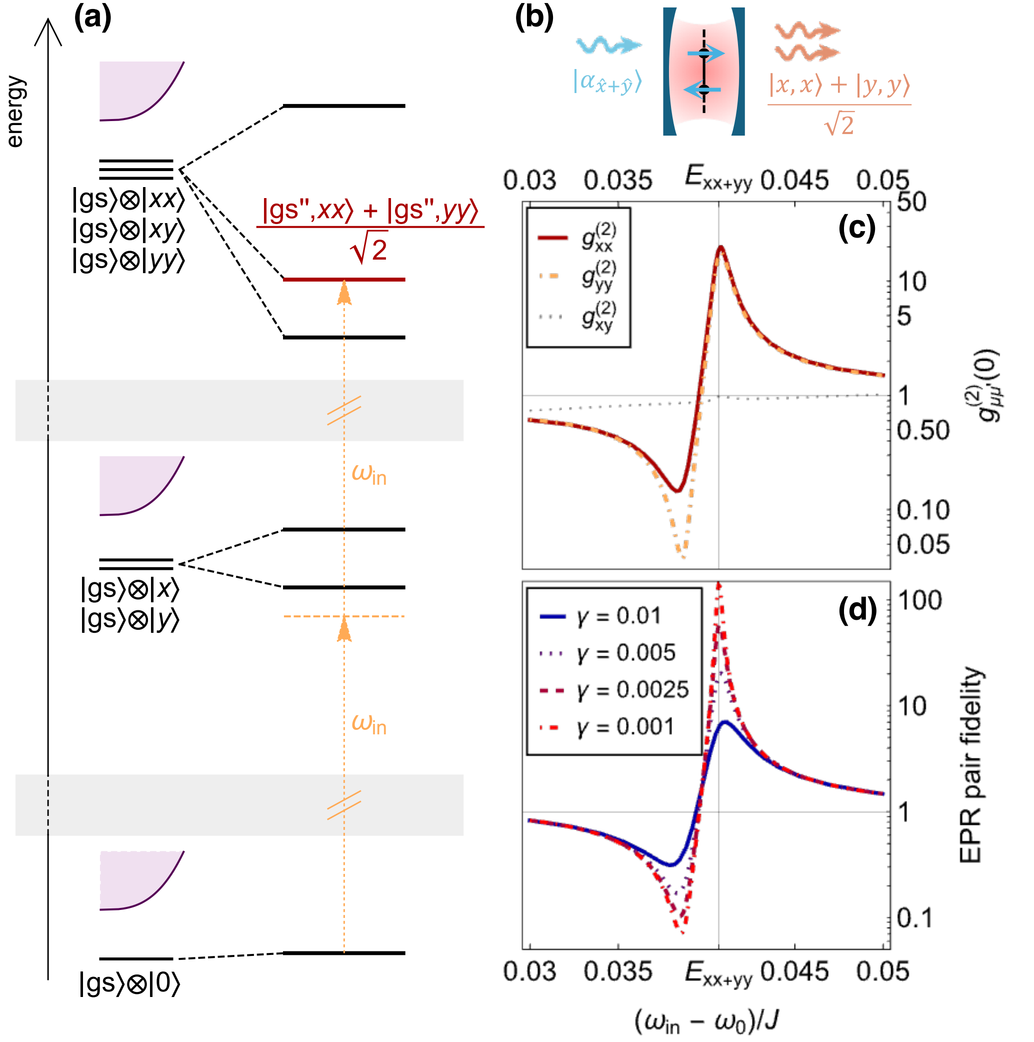
$$\begin{aligned} Z &= \int \mathcal{D}[a_\pm]\ e^{i(S_+-S_-)}\langle a_+(0)|n\rangle\langle n|a_-(0)\rangle \\ &= \int \mathcal{D}[a_\pm]\ e^{i(S_+-S_-)}\frac{1}{n!}\frac{\partial^n}{\partial u^n}e^{u(\bar{a}_+(0)a_-(0))}\Big|_{u=0} \\ &= \frac{1}{n!}\frac{\partial^n}{\partial u^n}\Big[\underbrace{\int \mathcal{D}[a_\pm]\ e^{i(S_+-S_--iu(\bar{a}_+(0)a_-(0)))}}_{\widetilde{Z}}\Big]\Big|_{u=0} \end{aligned}$$

All that changes is a new quadratic term in the action

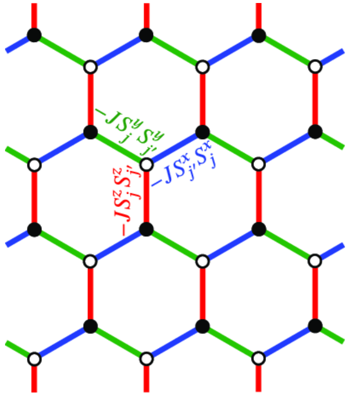
Directions 1: Chiral Materials and Bell Pairs

- 2411.08964

- Classical input → quantum output
  - Entangled light generation
- Combination of spatial symmetries and polarized light
  - Resonantly addressing a single eigenstate



Directions 2: Kitaev Spin Liquids



- Bunching/antibunching $g^{(2)}$
- Bond-bond correlations?
  - x, y, z bonds
  - Different linear polarizations should matter
  - Quantum output light (or input light?)