Frequency-Space Input-Output Relation

Let us consider the Hamiltonian for a single-mode cavity with bosonic mode a linearly coupled to a set of bosonic bath modes indexed by k

$$H = H_{\text{cav}} + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sqrt{\gamma_k}(a^\dagger b_k + a b_k^\dagger) \tag{1}$$

Now, the Heisenberg equation of motion for a single bath mode b_{k^*} is

$$\begin{aligned}
\partial_t b_{k^*} = -i[b_{k^*}, H] &= -i[b_{k^*}, H_{\text{cav}}] - i\sum_k \omega_k [b_{k^*}, b_k^\dagger b_k] - i\sum_k \sqrt{\gamma_k}[b_{k^*}, a^\dagger b_k + a b_k^\dagger] && (2)\\
&= -i\omega_{k^*}[b_{k^*}, b_{k^*}^\dagger b_{k^*}] - i\sqrt{\gamma_{k^*}}[b_{k^*}, a b_{k^*}^\dagger] && (3)\\
&= -i\omega_{k^*} b_{k^*} - i\sqrt{\gamma_{k^*}}a && (4)
\end{aligned}$$

where in the last line we used the commutation identities for bosons.
This can then be integrated to obtain

$$b_k(t, t_i) = e^{-i\omega_k(t-t_i)} b_k(t_i) - i\sqrt{\gamma_k}\int_{t_i}^t d\tau\ e^{-i\omega_k(t-\tau)} a(\tau) \tag{5}$$

we can check that this is correct by using the Leibniz integral rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} d\tau\ f(t,\tau) = f(t, b(t))\frac{d}{dt}b(t) - f(t, a(t))\frac{d}{dt}a(t) + \int_{a(t)}^{b(t)} d\tau \frac{\partial}{\partial t} f(t,\tau) \tag{6}$$

so we see that

$$\begin{aligned}
\partial_t b_{k^*} &= -i\omega_{k^*} e^{-i\omega_k(t-t_i)} b_{k^*}(t_i) - i\sqrt{\gamma_{k^*}}\left[e^{-i\omega_{k^*}(t-t)}a(t) - 0 + \int_{t_i}^t d\tau\ \ - i\omega_{k^*} e^{-i\omega_{k^*}(t-\tau)} a(\tau)\right] && (7)\\
&= -i\omega_{k^*}\left[e^{-i\omega_k(t-t_i)} b_{k^*}(t_i) - i\sqrt{\gamma_{k^*}}\int_{t_i}^t d\tau\ e^{-i\omega_{k^*}(t-\tau)} a(\tau)\right] - i\sqrt{\gamma_{k^*}}a(t) && (8)\\
&= -i\omega_{k^*} b_{k^*}(t, t_i) - i\sqrt{\gamma_{k^*}}a(t) && (9)
\end{aligned}$$

which recovers the equation of motion Eq. 4.
Now, introducing

$$\begin{aligned}
b_k^{\text{out}} &= b_k(t, t_i)e^{i\omega_k t} && (10)\\
b_k^{\text{in}} &= b_k(t_i, t_i)e^{i\omega_k t_i} && (11)
\end{aligned}$$

so returning to Eq. 5,

$$b_k^{\text{out}} e^{-i\omega_k t} = e^{-i\omega_k t} b_k^{\text{in}} - i\sqrt{\gamma_k}\int_{t_i}^t d\tau\ e^{-i\omega_k(t-\tau)} a(\tau) \tag{12}$$

which we can re-package as

$$\boxed{b_k^{\text{out}} = b_k^{\text{in}} - i\sqrt{\gamma_k}\int_{t_i}^t d\tau\ e^{i\omega_k \tau} a(\tau)} \tag{13}$$

which is the mode frequency space input-output relation.

Fourier Time Input-Output Relation

Now, to appreciate what we have done in the last section in a more physical context we should replace all t's with t_f a final time of detection/output. This gives us the input-output relation

$$b_k^{\text{out}} = b_k^{\text{in}} - i\sqrt{\gamma_k} \int_{t_i}^{t_f} d\tau \ e^{i\omega_k \tau} a(\tau) \tag{14}$$

with

$$b_k^{\text{out}} = b_k(t_f, t_i) e^{i\omega_k t_f} \tag{15}$$

$$b_k^{\text{in}} = b_k(t_i, t_i) e^{i\omega_k t_i} \tag{16}$$

Now let us introduce the *Fourier time* t so that the input/output operators in Fourier time are the (discrete) Fourier transform

$$b_k^{\text{in}}(t) = \sum_k \frac{\delta\omega_k}{2\pi} e^{-i\omega_k t} b_k^{\text{in}} \tag{17}$$

$$b_k^{\text{out}}(t) = \sum_k \frac{\delta\omega_k}{2\pi} e^{-i\omega_k t} b_k^{\text{out}} \tag{18}$$

where $\delta\omega$ is formally $2\pi/L$ for some high frequency cutoff L, but for us will be a uniform spacing between energy levels in the bath with the continuum limit $L \to \infty$ taken. We can then Fourier transform to find

$$b_k^{\text{out}}(t) = b_k^{\text{in}}(t) - i \sum_k \frac{e^{-i\omega_k t}}{L} \sqrt{\gamma_k} \int_{t_i}^{t_f} d\tau \ e^{i\omega_k \tau} a(\tau) \tag{19}$$

or with

$$\Gamma(t) = \sum_k \frac{e^{-i\omega_k t}}{L} \sqrt{\gamma_k} \tag{20}$$

by switching the order of the sum and the integral, we find

$$b_k^{\text{out}}(t) = b_k^{\text{in}}(t) - i \int_{t_i}^{t_f} d\tau \ \Gamma(t-\tau) a(\tau) \tag{21}$$

where if t_i is in the distant past and if t_f is in the distant future

$$\boxed{b_k^{\text{out}}(t) = b_k^{\text{in}}(t) - i \int_{-\infty}^{\infty} d\tau \ \Gamma(t-\tau) a(\tau)} \tag{22}$$