Analyzing Circuits with Impedances

Ohm's Law:

$$V = IZ \tag{1}$$

Where:[1]

$$Z_R = R \tag{2}$$
$$Z_L = i\omega L \tag{3}$$
$$Z_C = 1/i\omega C \tag{4}$$

In series:

$$Z = Z_1 + Z_2 + \cdots + Z_n \tag{5}$$

In parallel:

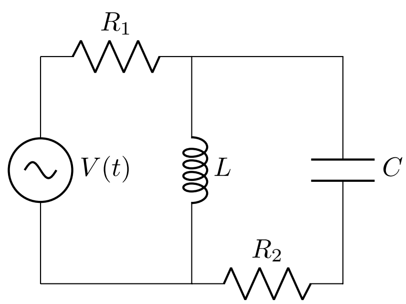
$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n} \tag{6}$$

Example: Amplitudes when driven by voltage

When we are only interested in voltage and current amplitudes and not phases, we use:

$$Z \quad \mapsto \quad |Z| = \sqrt{[\mathrm{Re}(Z)]^2 + [\mathrm{Im}(Z)]^2} \tag{7}$$

Consider the circuit below with $V(t) = V_0 \cos(\omega t)$:



What is the impedance of everything to the right of the battery? Express in the form $Z = R + i\chi$ for real R and χ.

Using series and parallel rules followed by complex arithmetic:[2]

$$\begin{aligned}
Z &= Z_{R_1} + \frac{1}{\frac{1}{Z_L} + \frac{1}{Z_C + Z_{R_2}}} &(8)\\
&= Z_{R_1} + \frac{Z_L(Z_C + Z_{R_2})}{Z_C + Z_{R_2} + Z_L} &(9)\\
&= R_1 + \frac{i\omega L((1/i\omega C) + R_2)}{(1/i\omega C) + R_2 + i\omega L} &(10)\\
&= R_1 + \frac{L/C + i(\omega L R_2)}{R_2 + i(\omega L - 1/\omega C)} \cdot \frac{R_2 - i(\omega L - 1/\omega C)}{R_2 - i(\omega L - 1/\omega C)} &(11)\\
&= R_1 + \frac{[LR_2/C + (\omega L R_2)(\omega L - 1/\omega C)] + i[(\omega L R_2^2) - (L/C)(\omega L - 1/\omega C)]}{R_2^2 + (\omega L - 1/\omega C)^2} &(12)\\
&= \left[R_1 + \frac{LR_2/C + (\omega L R_2)(\omega L - 1/\omega C)}{R_2^2 + (\omega L - 1/\omega C)^2}\right] + i\left[\frac{(\omega L R_2^2) - (L/C)(\omega L - 1/\omega C)}{R_2^2 + (\omega L - 1/\omega C)^2}\right] &(13)
\end{aligned}$$

[1] $Z = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z) = R + i\chi$ for resistance R and reactance χ.
[2] $i^2 = -1$ so $1/i = -i$

What is the maximum current amplitude through R_1?

We consider Ohm's law as $V^{\max} = I^{\max}|Z|$, with Z given by (13), with the current at its maximum amplitude when the voltage is at its maximum amplitude $V(t) = V_0$. Then the current through R_1 is the current that goes through the circuit as a whole and it given by:

$$I_{R_1}^{\max} = \frac{V_0}{|Z|} \tag{14}$$

$$= \frac{V_0}{\sqrt{\left[R_1 + \frac{LR_2/C+(\omega LR_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2 + \left[\frac{(\omega LR_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2}} \tag{15}$$

What is the maximum voltage amplitude through R_2?

By charge conservation, we know that the current that flows through the inductor plus the current that flows through R_2 is equal to the current that flows through R_1, so we have:

$$I_{R_1} = I_L + I_{R_2} \qquad \Longrightarrow \qquad I_{R_1}^{\max} = I_L^{\max} + I_{R_2}^{\max} \tag{16}$$

Additionally, since energy is single-valued, the voltage drop across the inductor must equal the voltage drop across the capacitor and R_2, so:

$$I_L Z_L = I_{R_2}(Z_C + Z_{R_2}) \tag{17}$$

Now, working with magnitudes, $I_{R_1} = I_{R_1}^{\max}$ as in (14-15), and $|Z_L| = \omega L$, and $|Z_C + Z_{R_2}| = \sqrt{R_2^2 + 1/(\omega C)^2}$, so we have two equations and two unknowns, solving we find:

$$I_L^{\max} = \frac{|Z_C + Z_{R_2}|^2}{|Z_L| + |Z_C + Z_{R_2}|} I_{R_1}^{\max} \tag{18}$$

$$I_{R_2}^{\max} = \frac{|Z_L| \cdot |Z_C + Z_{R_2}|}{|Z_L| + |Z_C + Z_{R_2}|} I_{R_1}^{\max} \tag{19}$$

Or, substituting in:

$$I_L^{\max} = \frac{R_2^2 + 1/(\omega C)^2}{\omega L + \sqrt{R_2^2 + 1/(\omega C)^2}} \cdot \frac{V_0}{\sqrt{\left[R_1 + \frac{LR_2/C+(\omega LR_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2 + \left[\frac{(\omega LR_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2}} \tag{20}$$

$$I_{R_2}^{\max} = \frac{\omega L\sqrt{R_2^2 + 1/(\omega C)^2}}{\omega L + \sqrt{R_2^2 + 1/(\omega C)^2}} \cdot \frac{V_0}{\sqrt{\left[R_1 + \frac{LR_2/C+(\omega LR_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2 + \left[\frac{(\omega LR_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2}} \tag{21}$$

Thus, the maximum voltage amplitude through R_2 is:

$$V_{R_2}^{\max} = I_{R_2}^{\max} R_2 \tag{22}$$

with $I_{R_2}^{\max}$ as in (21).

What is the phase difference between the voltage in R_1 and R_2?

Since both components are resistors, this is equivalent to asking the phase difference between the currents. Now, we cannot use $I_{R_1}^{\max}$ and $I_{R_2}^{\max}$ since these are magnitudes with no phase information.

We have:

$$I_{R_1}(t) = \frac{V_0 \cos(\omega t)}{Z} \tag{23}$$

$$= \frac{V_0 \cos(\omega t)}{\left[R_1 + \frac{LR_2/C+(\omega LR_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2 + \left[\frac{(\omega LR_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2} \tag{24}$$

$$\times \left(\left[R_1 + \frac{LR_2/C + (\omega LR_2)(\omega L - 1/\omega C)}{R_2^2 + (\omega L - 1/\omega C)^2}\right] - i\left[\frac{(\omega LR_2^2) - (L/C)(\omega L - 1/\omega C)}{R_2^2 + (\omega L - 1/\omega C)^2}\right]\right) \tag{25}$$

In general, the phase is:

$$\theta(f(t)) = \tan^{-1}\left(\frac{\mathrm{Im}(f(t))}{\mathrm{Re}(f(t))}\right) \tag{26}$$

So, we see:

$$\begin{aligned}
\theta_{R_1} &= \tan^{-1}\left(-\frac{\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}}{R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}}\right) &(27)\\
&= \tan^{-1}\left(-\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_1(R_2^2+(\omega L-1/\omega C)^2)+LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}\right) &(28)
\end{aligned}$$

Now:

$$\begin{aligned}
I_{R_2}(t) &= \frac{Z_L(Z_C+Z_{R_2})}{Z_L+Z_C+Z_{R_2}} I_{R_1}(t) &(29)\\
&= \frac{[LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]+i[\omega L R_2^2-(L/C)(\omega L-1/\omega C)]}{R_2^2+(\omega L-1/\omega C)^2} I_{R_1}(t) &(30)\\
&= \frac{[LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]+i[\omega L R_2^2-(L/C)(\omega L-1/\omega C)]}{1} &(31)\\
&\times \frac{1}{R_2^2+(\omega L-1/\omega C)^2}\frac{V_0\cos(\omega t)}{\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2+\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2} &(32)\\
&\times \left(\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]-i\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]\right) &(33)\\
&= \frac{1}{R_2^2+(\omega L-1/\omega C)^2}\frac{V_0\cos(\omega t)}{\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2+\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]^2} &(34)\\
&\times \Bigg\{\Bigg([LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right] &(35)\\
&\quad +[\omega L R_2^2-(L/C)(\omega L-1/\omega C)]\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]\Bigg) &(36)\\
&+ i\Bigg([\omega L R_2^2-(L/C)(\omega L-1/\omega C)]\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right] &(37)\\
&- [LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]\Bigg)\Bigg\} &(38)
\end{aligned}$$

So:

$$\theta_{R_2} = \tan^{-1}\left(\frac{[\omega L R_2^2-(L/C)(\omega L-1/\omega C)]\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]-[LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]}{[LR_2/C+(\omega L R_2)(\omega L-1/\omega C)]\left[R_1+\frac{LR_2/C+(\omega L R_2)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]+[\omega L R_2^2-(L/C)(\omega L-1/\omega C)]\left[\frac{(\omega L R_2^2)-(L/C)(\omega L-1/\omega C)}{R_2^2+(\omega L-1/\omega C)^2}\right]}\right) \tag{39}$$

If we have a 1 kHz signal with both resistors of 10 Ω, a capacitor of 1000 μF, and a 1 mH inductor, then:

$$\begin{aligned}
\theta_{R_1} &= -19.37^\circ &(40)\\
\theta_{R_2} &= 38.24^\circ &(41)
\end{aligned}$$

where V_{R_1} lags the voltage by 19.37°, and V_{R_2} leads the voltage by 38.24°, for a phase difference of:

$$|\theta_{R_1}-\theta_{R_2}| = 57.61^\circ \tag{42}$$