Electromagnetism

Field Strength
$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ F_{i0} = E_i, \quad F_{0i} = -E_i \\ F_{ij} = \varepsilon_{ijk} B^k \end{cases}$$

Maxwell
$$\begin{cases} \partial_\mu F^{\nu\mu} = J^\nu \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \end{cases}$$

Lorentz
$$f^\alpha = e\, \eta_{\beta\gamma} F^{\alpha\beta} \frac{d\xi^\gamma}{d\tau}$$

Newton
$$m \frac{d^2 \xi^\alpha}{d\tau^2} = f^\alpha$$

continuity
$$\partial_\nu J^\nu = 0$$

Mechanics; Stress-Energy

4-velocity / momenta
$$u^\alpha = \frac{d\xi^\alpha}{d\tau} \; ; \; p^\alpha = m u^\alpha$$

$$u^\alpha u_\alpha = -1$$

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P \eta^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$u^\mu \nabla_\mu u^\nu = 0$$

$$\nabla_\mu T^{\mu\nu} = 0$$

Math Identities

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2} + \frac{3}{8}x^2 - \dots$$

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right)$$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, V^\mu \right)$$

$$g = \det(g_{\mu\nu}) \neq \det(g^{\mu\nu})$$

Levi-Civita

2D

$$\varepsilon_{ij} \varepsilon^{kl} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k$$

$$\varepsilon_{ij} \varepsilon^{jk} = \delta_i^k \; ; \; \varepsilon_{ij} \varepsilon^{ij} = 2$$

3D

$$\varepsilon_{ijk} \varepsilon^{lmk} = \delta_i^l \delta_j^m - \delta_i^m \delta_j^l$$

$$\varepsilon_{ijk} \varepsilon^{ljk} = 2 \delta_i^l$$

$$\varepsilon_{ijk} \varepsilon^{ijk} = 6$$

Metric

$$[\eta] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$

$$\delta^\mu{}_\mu = D$$

"trace = contract w/ metric"

$$\underbrace{d\tau}_{} = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

ds if not rest frame

Changing variables & frames

chain rule

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x'^\beta} dx'^\beta$$

$$\frac{dx^\alpha}{dt} = \frac{dx^\alpha}{dx'^\beta}\frac{dx'^\beta}{dt}$$

identities

$$\frac{\partial x^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial x^\nu} = \delta^\mu{}_\nu$$

$$\frac{\partial^2 x^\mu}{\partial x^\nu \partial x^\rho} = 0$$

examples

$$A^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} A^\mu$$

$$B_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} B_\mu$$

Christoffel

geodesic

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)$$

$$= \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} + g_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}$$

$$\Gamma^\lambda_{00} = -\frac{1}{2} g^{\lambda\nu}\frac{\partial g_{00}}{\partial x^\nu}$$

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\mu\rho}$$

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma^\lambda_{\mu\nu} - \frac{\partial x^\lambda}{\partial x^{\nu'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^\lambda \partial x^\mu}$$

Covariant derivatives

$$\nabla^\mu AB = (\nabla^\mu A)B + A(\nabla^\mu B)$$

η & g can pass through ∇^μ i.e. $\nabla_\lambda g_{\mu\nu} = 0$

$$\nabla^\mu(\text{scalar}) = \partial^\mu(\text{scalar})$$

$\nabla_\mu V^\nu$ transforms as a tensor

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda$$

$$\nabla^2 := g^{\mu\nu}\nabla_\mu\nabla_\nu \; ; \; \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

$$[\nabla_\mu, \nabla_\nu] A^\alpha = R^\alpha{}_{\sigma\mu\nu} A^\sigma$$

$$[\nabla_\mu, \nabla_\nu] A_\alpha = -R^\sigma{}_{\alpha\nu\mu} A_\sigma$$

$$[\nabla_\kappa, \nabla_\nu] T^\lambda{}_\mu = T^\sigma{}_\mu R^\lambda{}_{\sigma\nu\kappa} - T^\lambda{}_\sigma R^\sigma{}_{\mu\nu\kappa}$$

Equivalence Principle

Small enough region → freely falling as in S. Rel.

$$\frac{d^2\xi_\alpha}{d\tau^2} = 0 \text{ and } d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$$

freely falling (rest frame) $\Rightarrow ds^2 \mapsto d\tau^2$

General Covariance Principle

If an eq. holds w/o gravity and is generally covariant → then it holds in a general g field

same form by coord. system

$$\partial_\mu \mapsto \nabla_\mu, \; \eta \to g$$

The Enstein Field Equation and Related Quanties

Riemann & Ricci

$$R^{\lambda}{}_{\mu\kappa\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \longleftarrow R_{\mu\nu} = R_{\nu\mu}, \quad g^{\alpha\beta}R_{\beta\mu\alpha\nu}$$

$$R = g^{\mu\nu}R_{\mu\nu}$$

$$R_{\lambda\mu\kappa\nu} = g_{\lambda\eta}R^{\eta}{}_{\mu\kappa\nu} \longleftarrow R_{\blacksquare\square} = R_{\square\blacksquare}$$

and antisymmetric in 1,2 & 3,4

$$= \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\mu}\partial x^{\kappa}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\lambda}\partial x^{\kappa}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\lambda}\partial x^{\nu}}\right) + g_{\alpha\beta}\left(\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\beta}_{\mu\kappa} - \Gamma^{\alpha}_{\kappa\lambda}\Gamma^{\beta}_{\mu\nu}\right)$$

cyclicity & Bianchi

$$R_{\lambda\mu\kappa\nu} + R_{\lambda\nu\mu\kappa} + R_{\lambda\kappa\nu\mu} = 0$$

$$\nabla_{\eta}R_{\lambda\mu\kappa\nu} + \nabla_{\nu}R_{\lambda\mu\eta\kappa} + \nabla_{\kappa}R_{\lambda\mu\nu\eta} = 0$$

$$C_{\lambda\mu\kappa\nu} = R_{\lambda\mu\kappa\nu} - \frac{1}{D-2}\left(g_{\kappa\lambda}R_{\mu\nu} - g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\kappa}R_{\lambda\nu} + g_{\mu\nu}R_{\lambda\kappa}\right) + \frac{R}{(D-1)(D-2)}\left(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\kappa}\right)$$

$\mathrm{Tr}(C) = 0 \rightsquigarrow$ traceless in all indices

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \longleftarrow G_{\mu\nu} = G_{\nu\mu}, \quad T_{\mu\nu} = T_{\nu\mu}$$

$\nabla_{\mu}G^{\mu\nu} = 0$ since $\nabla_{\mu}T^{\mu\nu} = 0$

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

"cotton tensor"

$$\nabla^{\nu}C_{\lambda\mu\kappa\nu} = J_{\lambda\mu\kappa} = \frac{1}{2}\left(\nabla_{\lambda}R_{\kappa\mu} - \nabla_{\mu}R_{\kappa\lambda} + \frac{1}{6}\left(g_{\kappa\mu}\nabla_{\lambda}R - g_{\kappa\lambda}\nabla_{\mu}R\right.\right.$$

$$g_{\alpha\beta}(\xi) = \eta_{\alpha\beta} - \frac{1}{3}R_{\alpha\gamma\beta\delta}(\xi_0)(\xi^{\gamma} - \xi^{\gamma}_0)(\xi^{\delta} - \xi^{\delta}_0) - \dots$$

freely falling

$R_{\dots}$ vanishes $\Longleftrightarrow$ No gravitational field

$\Updownarrow$ $[\nabla_{\mu}, \nabla_{\nu}] = 0$

$\Updownarrow$ $\eta_{\mu\nu}$ everywhere

Weak-Field General Relativity

Newtonian Approx

$$g_{00} \approx -(1+2\Delta\Phi)$$

$$\nabla^2 \Phi = 4\pi G_N \rho \leftarrow T_{00}$$

$$\left|\frac{dt}{dt}\right| >> \left|\frac{dx^i}{dt}\right| \leftarrow \text{g-field quasistatic}$$

$$\Phi = -\frac{G_N M}{R} << 1$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$G_{00} \approx -\vec{\nabla}^2 g_{00}$$

einstein tensor

$$G_{\mu\nu} = -\frac{1}{2}\Big(\partial^2 h_{\mu\nu} - \eta_{\mu\nu}\partial^2 h - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\mu\alpha} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \partial_\mu \partial_\nu h\Big)$$

$$\partial^2 = \eta^{\mu\nu}\partial_\mu \partial_\nu$$

$$h = \eta^{\mu\nu} h_{\mu\nu}$$

gravitational waves

$$\tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu \varepsilon_\nu(x) + \partial_\nu \varepsilon_\mu(x) + O(\varepsilon^2)$$

"gauge invariant"

Choose $\partial^2 h_{\mu\nu} = \frac{1}{2}\partial_\nu h$ "harmonic gauge"

$$G_{\mu\nu} = -\frac{1}{2}\partial^2\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right)$$

whence

$$\partial^2 \underbrace{\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\right)}_{\mathcal{H}_{\mu\nu}} = -16\pi G_N T_{\mu\nu}$$

$\partial^\mu \mathcal{H}_{\mu\nu} = 0$, $T_{\mu\nu} = 0$ in vacuum in this gauge

$$\Rightarrow \partial^2 \mathcal{H}_{\mu\nu} = 0$$

↳ 2 gravitational waves

ACTION

$$S = \int dt\ \mathcal{L}(q,\dot{q},t)$$

Physical so'ns. are stationary $\delta S = 0$

Euler-Lagrange $\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$

$\mathcal{L}$ only defined up to a total derivative

EINSTEIN-HILBERT ACTION

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\ \sqrt{-g}\, R$$

$$S = S_{EH} + \int d^4x\ \sqrt{-g}\ \mathcal{L}_m(g,\psi)$$

# of examples

$$\nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\mu \phi\right)$$

$$\delta_g R_{\mu\nu} = \delta_g g^{\mu\nu} R_{\mu\nu} + \nabla_\mu\left(-\nabla_\nu \delta g^{\mu\nu} + g_{\alpha\beta}\nabla^\mu \delta g^{\alpha\beta}\right)$$

$$\delta_g \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\delta_g g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta}\,\delta g^{\alpha\beta}$$

$$\delta_\alpha \frac{1}{\alpha} = -\frac{1}{\alpha^2}\delta_\alpha \alpha$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\ \frac{\delta\left(\sqrt{-g}\,\mathcal{L}_m\right)}{\delta g^{\mu\nu}}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu G^{\mu\nu} = 0$$

THE SCHWARTSCHILD SOLUTION

$-\frac{8\pi G_N}{3}\Lambda r^2$

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\Omega^2$$

27 Oct $\overset{pg3}{\Rightarrow}$ expressions for $R_{\mu\nu}$

ISOMETRY: $\delta g_{\mu\nu} = 0$

$$\tilde{x}^\mu = x^\mu - \xi^\mu(x)$$

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

killing vectors ⟵ # of killing vectors = # of symmetries

29 oct $\overset{pg2}{\Rightarrow}$ 4 killing vectors for SCHWARTZSCHILD

$u^\mu u_\mu = -\varepsilon$ → L & E conserved quantities & V_{eff}

LIGHT BENDING (Nov 3)

↳ $$\Delta\varphi = 2\int_{R_0}^{\infty} \frac{dr}{r^2\sqrt{b^{-2} - \frac{1}{r^2} + \frac{r_s}{r^3}}}$$

MERCURY PERIHELION (Nov 5)

$$\Delta\varphi = \frac{6\pi G_N M}{c^2(1-e^2)a}$$

BLACK HOLES

Tortoise: $r_* = r + r_s \ln\left(\frac{r}{r_s} - 1\right)$

$$ds^2 = \left(\frac{r_s}{r} - 1\right)dt^2 - \frac{dr^2}{\frac{r_s}{r} - 1} + r^2 d\Omega^2 \quad \text{for } r < r_s$$

$$\frac{dt}{dr} = \pm \frac{1}{1-\frac{r_s}{r}}$$

outgoing ($+$), ingoing ($-$)

GRAVITATIONAL WAVES (19 Nov)

WEAK FIELD

$$\partial^2 H_{\mu\nu} = -16\pi G_N T_{\mu\nu}$$

$$H_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik_\lambda x^\lambda} + h.c.$$

$H_{\mu\nu} = h_{\mu\nu}$ in a good gauge

Transverse traceless gauge $\begin{cases} \partial^\mu h_{\mu\nu} = 0 \\ h^\mu{}_\mu = 0 \\ h_{0\mu} = 0 \end{cases}$

2 polarizations

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos(\omega(t-z))$$

$$ds^2 = -dt^2 + (1+h_+\cos\varphi)dx^2 + (1-h_+\cos\varphi)dy^2 + 2h_\times \cos\varphi\, dx dy + dz^2$$

No effect on a single particle

$$\Delta l \simeq \left(1 + \frac{h_+}{2}\cos\omega t\right)\Delta x$$

$$\Delta l \simeq \left(1 - \frac{h_+}{2}\cos\omega t\right)\Delta y$$

Generation w/ two rotating bodies $\Rightarrow$ 24 Nov

COSMOLOGY

Homog & isotropic

$$r(t) = a(t)\, r_0$$

$$v(t) = \underbrace{\frac{\dot a}{a}}_{H} r(t)$$

FLRW

$$ds^2 = -dt^2 + a^2(t)\, g_{ij}(q)\, dx^i dx^j$$

$$= -dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$$

FRIEDMAN

$$\frac{\dot a^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G_N}{3} \rho(t)$$

$$\frac{\ddot a}{a} = -\frac{4\pi G_N}{3}(\rho + 3P)$$

$$\rho_{dust} = \frac{\alpha_m}{a^3}$$

$$\rho_{rad} = \frac{\alpha_r}{a^4}$$

$$\rho_\Lambda = -\Lambda$$

$$\rho_{inf} = +\Lambda$$

$$a \sim t^{\frac{2}{3(1+w)}} \quad \text{for } P = w\rho$$

REDSHIFT & Z

$$\frac{d^2 r}{dt^2} = \frac{d}{dr}\left(\frac{dr}{dt}\right) \cdot \frac{dr}{dt}$$