Abstract

Here I pedagogically reiterate the work developed in PNAS **112**, 12645 (2015) and 2304.02344, with a reference to 1305.06412.

Microscopic Model

Let there be M conserved charges Q_m occupying a M-railed quasi-1D system. The system is quasi-1D in the sense that particles of species m can only hop along and not between rails, but the probability of hopping is dependent on the number of nearby particles on other rails m'. Now, let the conserved charges be given by

$$Q_m = \sum_{l=1}^{L} n_m^l \tag{1}$$

the total number of particles of species m on the m'th rail. Let this be an exclusion problem so that $n_m^l = 1$ or 0 so that sites may either be singly occupied or unoccupied. Now consider totally asymmetric (TASEP) hopping with periodic boundary conditions. Hopping is stroboscopic in time and random. In one time step, consider site l of rail m and select a random variable r from $(0,1)$:

$$\begin{cases} \text{hop,} & r > r_m^l \text{ and } n_m^l(1 - n_m^{l+1}) = 1 \\ \text{stay,} & r \leq r_m^l \text{ or } n_m^l(1 - n_m^{l+1}) = 0 \end{cases} \tag{2}$$

where

$$r_m^l(t) = r_m + \sum_{m' \neq m} \Gamma_{mm'}(n_{m'}^l(t) + n_{m'}^{l+1}(t)) \tag{3}$$

is the threshold for hopping to occur. Note that Γ couples rail m to the other rails $m' \in \{1, \ldots, M\}$. From this we obtain the current over the whole rail

$$j_m = \rho_m(1 - \rho_m)\left(r_m + \sum_{m' \neq m} \Gamma_{mm'}\rho_{m'}\right) \tag{4}$$

where $\rho_m = Q_m/L$ is the *average* density. We now introduce the "compressability matrix"

$$K_{mm'} = \frac{1}{L}\sum_{l=1}^{L}\langle(n_m^l - \frac{Q_m}{L})(n_{m'}^l - \frac{Q_{m'}}{L})\rangle \tag{5}$$

where the average is evaluated over different realizations through time. With no exchange between rails K becomes diagonal and has diagonal entries

$$K_{mm} = \frac{1}{L}\sum_{l=1}^{L}\langle(n_m^l - \rho_m)^2\rangle \tag{6}$$

$$= \frac{1}{L}\sum_{l=1}^{Q_m}\langle(1 - \rho_m)^2\rangle + \frac{1}{L}\sum_{l=1}^{L-Q_m}\langle\rho_m^2\rangle \tag{7}$$

$$= \rho_m(1 - \rho_m)^2 + (1 - \rho_m)\rho_m^2 \tag{8}$$

$$= \rho_m(1 - \rho_m) \tag{9}$$

this expression will be useful later.

Hydrodynamic Model

We then have the continuity equation

$$\partial_t \rho_m(t,l) + \partial_l j_m(t,l) = 0 \tag{10}$$

where we have taken the continuum limit by introducing the density in some region of length Δl_m with Δn_m particles: $\rho_m(t,l) = \Delta n_m(t)/\Delta l_m$, and the particulate current $j_m(t,l) = v_m(t,l)\rho_m(t,l)$ for a velocity distribution $v_m(t,l)$. We are free to use the chain rule

$$\frac{\partial j_m}{\partial l} = \frac{\partial j_m}{\partial \rho_m} \frac{\partial \rho_m}{\partial l} \tag{11}$$

and doing so obtain

$$\partial_t \rho_m(t,l) + \partial_{\rho_m} j_m \; \partial_l \rho_m(t,l) = 0. \tag{12}$$

Now, with no microscopic underpinning, one *assumes* that there is a diffusion term

$$-D_m \partial_{ll} \rho_m(t,l) \tag{13}$$

and a (fluctuating/stochastic) white noise term

$$N_m \partial_l w_m \tag{14}$$

where D_m and N_m set the amplitude of the diffusion an the noise terms and w_m is white noise term. With the inclusion of these terms we have the "non-linear flucuating hydrodynamic" equation

$$\partial_t \rho_m(t,l) + \partial_{\rho_m} j_m \; \partial_l \rho_m(t,l) - D_m \partial_{ll} \rho_m(t,l) + N_m \partial_l w_m = 0. \tag{15}$$

Now take the limit of small fluctuations p so that

$$\rho_m(t,l) = \rho_m + p_m(t,l) \tag{16}$$

where high orders in p can be neglected as small and $\rho_m = Q_m/L$ is the average charge density of rail m. Doing so we find

$$\partial_t p_m(t,l) + r_m(1 - 2\rho_m)\partial_l p(t,l) - r_m \partial_l[p(t,l)^2] - D_m \partial_{ll} p_m(t,l) + N_m \partial_l w_m = 0, \tag{17}$$

where we observed that for $j_m(t,l) := r_m \rho_m(t,l)(1 - \rho_m(t,l))$

$$\partial_\rho j_m(t,l) \; \partial_l \rho_m(t,l) = r_m(1 - 2\rho_m(t,l))\partial_l \rho_m(t,l) \tag{18}$$
$$= \text{product rule shenanigans} \tag{19}$$
$$= r_m(1 - 2\rho_m)\partial_l p(t,l) - r_m \partial_l[p(t,l)^2]. \tag{20}$$

Now this is just a re-branding of the expression in the PNAS paper:

$$\partial_t p_\mu + J_{\mu\nu}\partial_l p^\nu + r_{\mu\nu\lambda}\partial_l p^\nu p^\lambda - D_{\mu\nu}\partial_{ll} p^\nu + N_{\mu\nu}\partial_l w^\nu = 0 \tag{21}$$

which we have written in covariant notation where the rails are $m \mapsto \mu$, $m' \mapsto \nu$ so that $p_m(t,l) \mapsto p_\mu$. Here the tensors are defined as

$$J_{\mu\nu} = \frac{\partial j_\mu}{\partial \rho^\mu} \tag{22}$$

$$r_{\mu\nu\lambda} = \frac{\partial^2 j_\mu}{\partial \rho^\nu \partial \rho^\lambda} \tag{23}$$

where the currents j_μ are obtained from the microscopic model, i.e. Eq. (4). $D_{\mu\nu}$ and $N_{\mu\nu}$ are put in by hand.

Change of Basis

Now let us consider the change of basis

$$\phi_\alpha = O_{\alpha\mu} p^\mu; \quad p_\mu = O_{\alpha\mu}^{-1} \phi^\alpha \tag{24}$$

which is chosen so that

$$O_{\alpha\mu} J^{\mu\nu} O_{\nu\alpha}^{-1} = V_{\alpha\beta} \tag{25}$$

where $V_{\alpha\beta}$ is diagonal so that only $V_{\alpha\alpha}$ are non-vanishing. We change from μ and ν to distinguish between the *physical* rails μ and the *normal mode* rails α. These diagonal elements are the eigenvalues of J which are the characteristic velocities of the normal modes. Having performed this diagonalization we have the hydrodynamic equation

$$\partial_t (O^{-1})_{\alpha\mu} \phi^\alpha + J_{\mu\nu} \partial_l (O^{-1})^{\alpha\nu} \phi_\alpha + r_{\mu\nu\lambda} \partial_l (O^{-1})^{\alpha\nu} \phi_\alpha (O^{-1})^{\beta\lambda} \phi_\beta - D_{\mu\nu} \partial_{ll} (O^{-1})^{\alpha\nu} \phi_\alpha + N_{\mu\nu} \partial_l w^\nu = 0 \tag{26}$$

or, applying with O_δ^μ and *assuming* the O's are constants relative to t and l

$$O_\delta^\mu (O^{-1})_{\alpha\mu} \partial_t \phi^\alpha + O_\delta^\mu J_{\mu\nu} (O^{-1})^{\alpha\nu} \partial_l \phi_\alpha + O_\delta^\mu r_{\mu\nu\lambda} (O^{-1})^{\alpha\nu} (O^{-1})^{\beta\lambda} \partial_l \phi_\alpha \phi_\beta \tag{27}$$

$$- O_\delta^\mu D_{\mu\nu} (O^{-1})^{\alpha\nu} \partial_{ll} \phi_\alpha + O_\delta^\mu N_{\mu\nu} \partial_l w^\nu = 0 \tag{28}$$

which reduces to

$$\partial_t \phi_\delta + \partial_l V_\delta^\alpha \phi_\alpha + \partial_l R_\delta^{\alpha\beta} \phi_\alpha \phi_\beta - \partial_{ll} \widetilde{D}_\delta^\alpha \phi_\alpha + \partial_l \widetilde{N}_{\delta\mu} w^\mu = 0 \tag{29}$$

where

$$\phi_\delta = O_\delta^\mu (O^{-1})_{\alpha\mu} \partial_t \phi^\alpha \tag{30}$$

$$V_\delta^\alpha = O_\delta^\mu J_{\mu\nu} (O^{-1})^{\alpha\nu} \tag{31}$$

$$R_\delta^{\alpha\beta} = O_\delta^\mu r_{\mu\nu\lambda} (O^{-1})^{\alpha\nu} (O^{-1})^{\beta\lambda} \tag{32}$$

$$\widetilde{D}_\delta^\alpha = O_\delta^\mu D_{\mu\nu} (O^{-1})^{\alpha\nu} \tag{33}$$

$$\widetilde{N}_{\delta\mu} = O_\delta^\mu N_{\mu\nu} \tag{34}$$

and we again assumed that the O's are constants relative to t and l.

Renormalization Group

Now we consider the renormalization group operator R_λ which acts as

$$\mathsf{R}_\lambda f(l,t) = \lambda^a f(l\lambda^b, t\lambda^c) \tag{35}$$

with exponents a, b, c, or as we will be interested in

$$\mathsf{R}_\lambda \phi(t,l) = \lambda^a \phi(t\lambda^c, l\lambda^b) \tag{36}$$

where $a = -1$ (see 2304.02344) and we can reexpress this as

$$\mathsf{R}_\lambda \phi(t,l) = \lambda^{-1} \phi(t\lambda^{yz}, l\lambda^{-y}). \tag{37}$$

For KPZ we have (see 2304.02344)

$$y = 2 \tag{38}$$
$$z = 3/2 \tag{39}$$

For diffusive (Edwards-Wilkinson) we have

$$y = 2 \tag{40}$$
$$z = 2 \tag{41}$$

where z is the "dynamical exponent".

Two-Point Correlation Function

Now, we are interested in the density-density correlation function

$$S_{\alpha\beta} = \langle \phi_\alpha(t,l)\phi_\beta(0,0) \rangle \tag{42}$$

to simplify the analysis one assumes the "hyperbolicity" condition that all normal mode velocities are different (J's eigenvalues are non-degenerate). Such an assumption is not needed (see 2304.02344), but doing so means that in the long-distance/long-time regime only the diagonal elements of S are non-vanishing. From page 11 of 1305.06412 we have

$$\partial_t S_{\alpha\alpha}(t,l) = -V_{\alpha\alpha}\partial_l S_{\alpha\alpha}(t,l) - D_{\alpha\alpha}\partial_{ll}S_{\alpha\alpha}(t,l) + \int_0^t d\tau \int_{-\infty}^\infty d\ell\ S_{\alpha\alpha}(t-\tau, l-\ell)\partial_{\ell\ell}M_{\alpha\alpha}(\tau,\ell) \tag{43}$$

which is obtained from (extensive) manipulations of Eq. (29), where

$$M_{\alpha\alpha}(\tau,\ell) = R_{\alpha\beta\gamma}R_{\alpha\beta\gamma}S_\beta(\tau,\ell)S_\gamma(\tau,\ell) \tag{44}$$

is known as the "memory kernel". I am being sloppy with indices here–everything here is written covariant while this is not actually the case, and only the β and γ indices are contracted. This is now a system of equations that can be solved.

Solution

Let the Fourier transform be

$$\widehat{f}(k) = \int_{-\infty}^\infty dl\ e^{-ikl}f(l) \tag{45}$$

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and the Laplace transform be

$$\widetilde{f}(\omega) = \int_0^\infty dt \; e^{-\omega t} f(t) \tag{46}$$

First, let us note that under Fourier transform

$$\partial_l \mapsto -ik \tag{47}$$

so that

$$-V_{\alpha\alpha}\partial_l - D_{\alpha\alpha}\partial_{ll} \mapsto ikV_{\alpha\alpha} + k^2 D_{\alpha\alpha} \tag{48}$$

where for the time being we ignore what object (the two-point correlation function) these act on. Additionally we can Laplace transform using the identity

$$\partial_t f(t) \mapsto \omega \widetilde{f}(\omega) - \widetilde{f}(0) \tag{49}$$

so that

$$\partial_t S_{\alpha\alpha}(t,l) \mapsto \omega \widetilde{S}_{\alpha\alpha}(\omega,l) - \widetilde{S}_{\alpha\alpha}(0,l) \tag{50}$$

where $\widetilde{S}_{\alpha\alpha}(0,l) = 1$ so that we can write (hereafter I am being schematic rather than rigorous)

$$1 = \omega \widetilde{S}_{\alpha\alpha}(\omega,l) + ikV_{\alpha\alpha}S_{\alpha\alpha} + k^2 D_{\alpha\alpha}S_{\alpha\alpha} + \int_0^\infty d\omega e^{-\omega \bar{t}} \int_{-\infty}^\infty d\bar{k} \; S_{\alpha\alpha}k^2 \sum_{\beta\gamma} R_{\alpha\beta\gamma}R_{\alpha\beta\gamma}S_\beta(\bar{t},k)S_\gamma(\bar{t},k-\bar{k})$$

$$\tag{51}$$

which becomes

$$\widetilde{S}_{\alpha\alpha}(\omega,l) = \frac{1}{\omega + ikV_{\alpha\alpha} + k^2 D_{\alpha\alpha} + \int_0^\infty d\omega e^{-\omega \bar{t}} \int_{-\infty}^\infty d\bar{k} \; k^2 \sum_{\beta\gamma} R_{\alpha\beta\gamma}R_{\alpha\beta\gamma}S_\beta(\bar{t},k)S_\gamma(\bar{t},k-\bar{k})} \tag{52}$$

Now, we will be interested in the long-distance (small-k) long-time behavior. Since the velocities are different we assume "strict hyperbolicity" which means that $R_{\alpha\beta\gamma} = 0$ when $\beta \neq \gamma$ and we have

$$\widetilde{S}_{\alpha\alpha}(\omega,l) = \frac{1}{\omega + ikV_{\alpha\alpha} + k^2 D_{\alpha\alpha} + k^2 \sum_\beta (R_{\alpha\beta\beta})^2 \int_0^\infty d\omega e^{-\omega \bar{t}} \int_{-\infty}^\infty d\bar{k} \; S_\beta(\bar{t},k)S_\beta(\bar{t},k-\bar{k})} \tag{53}$$

Now since we are interested in the small-p limit, we want to know what the dominant exponent z_a so that $\widetilde{S}_{\alpha\alpha}(\omega,k) = k^{-z_\alpha}f_\alpha(\omega)$. Multiplying on the top and bottom by k^{-z_α} we have

$$\widetilde{S}_{\alpha\alpha}(\omega,k) = \frac{k^{-z_\alpha}}{\omega k^{-z_\alpha} + ik^{1-z_\alpha}V_{\alpha\alpha} + k^{2-z_\alpha}D_{\alpha\alpha} + k^{2-z_\alpha}\sum_\beta (R_{\alpha\beta\beta})^2 \int_0^\infty d\omega e^{-\omega \bar{t}} \int_{-\infty}^\infty d\bar{k} \; S_\beta(\bar{t},k)S_\beta(\bar{t},k-\bar{k})}$$

$$\tag{54}$$

Introducing $\zeta_\alpha = (\omega + ikV_{\alpha\alpha})|k|^{-z_\alpha}$ and $Q_{\alpha\beta} = (R_{\alpha\beta\beta})^2 \int_0^\infty d\omega e^{-\omega \bar{t}} \int_{-\infty}^\infty d\bar{k} \; S_\beta(\bar{t},k)S_\beta(\bar{t},k-\bar{k})$

$$\widetilde{S}_{\alpha\alpha}(\omega,k) = \frac{k^{-z_\alpha}}{\zeta_\alpha + k^{2-z_\alpha}D_{\alpha\alpha} + k^{2-z_\alpha}Q_{\alpha\alpha} + k^{2-z_\alpha}\sum_{\beta\neq\alpha}Q_{\alpha\beta}} \tag{55}$$

Next we note that

$$(k^{2-z_\alpha})^2 = (k^{-z_\alpha})^{2-z_\alpha-1/z_\alpha}k^{3-2z_\alpha} \sim ((\omega + ikV_{\alpha\alpha})k^{-z_\alpha})^{2-z_\alpha-1/z_\alpha}k^{3-2z_\alpha}\zeta_\alpha^{2-z_\alpha-1/z_\alpha}k^{3-2z_\alpha} \tag{56}$$

After a similar manipulation on the $\alpha \neq \beta$ term we obtain the final expression

$$\widetilde{S}_{\alpha\alpha}(\omega, k) = \frac{k^{-z_\alpha}}{\zeta_\alpha + k^{2-z_\alpha} D_{\alpha\alpha} + Q_{\alpha\alpha}\zeta_\alpha^{2-z_\alpha-1/z_\alpha} k^{3-2z_\alpha} + \sum_{\beta\neq\alpha} Q_{\alpha\beta}(-i(V_{\alpha\alpha} - V_{\beta\beta})^{1/z_\beta-1} k^{1-z_\alpha+1/z_\beta}}$$

(57)

where for simplicity we have assumed that $k > 0$. There are now a few regimes depending on the structure of Q and therefore R. If $R_{\alpha\beta} = 0$ for all R we get diffusive behavior

$$\widetilde{S}_{\alpha\alpha}(\omega, k) = \frac{k^{-z_\alpha}}{\zeta_\alpha + k^{2-z_\alpha} D_{\alpha\alpha}}$$

(58)

in particular $1 = k^{2-z_\alpha}$ is the power counting condition to find $z_\alpha = 2$. If only $R_{\alpha\alpha}$ are non-vanishing

$$\widetilde{S}_{\alpha\alpha}(\omega, k) = \frac{k^{-z_\alpha}}{\zeta_\alpha + k^{2-z_\alpha} D_{\alpha\alpha} + Q_{\alpha\alpha}\zeta_\alpha^{2-z_\alpha-1/z_\alpha} k^{3-2z_\alpha}}$$

(59)

and the dominant term is k^{3-2z_α} or, power-counting $z_\alpha = 3/2$ which is the KPZ class. The Fibonacci classes are then obtained from $R_{\alpha\beta}$ being non-vanishing.