Optical Absorption of Graphene in the Elliptic Dirac Cone Limit

Let the incident energy flux be (in Gaussian units):

$$W_{\text{incident}} = \frac{\omega^2}{4\pi c}|A|^2 \tag{1.1}$$

And the absorbed power be:

$$W_{\text{absorbed}} = \langle w \rangle \hbar\omega \tag{1.2}$$

Where Fermi's Golden rule gives the transition rate at low temperature ($E_f = \hbar\omega/2$ is the final energy):

$$w = \frac{2\pi}{\hbar}|\langle f|H'|i\rangle|^2 D(E_f) \tag{1.3}$$

Now for graphene in the elliptic Dirac-Cone limit with $\theta = \arg(v_x k_x + i v_y k_y)$:

$$\begin{aligned}
H_0 &= \hbar(v_x k_x \sigma_x + v_y k_y \sigma_y) && (1.4)\\
&= \hbar \begin{pmatrix} 0 & v_x k_x - i v_y k_y \\ v_x k_x + i v_y k_y & 0 \end{pmatrix} && (1.5)\\
&= \hbar\sqrt{(v_x k_x)^2 + (v_y k_y)^2} \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} && (1.6)
\end{aligned}$$

As above, we consider the perturbations:

$$\begin{aligned}
H'_x &= -\frac{e|A|}{\hbar c}\frac{\partial H_0}{\partial k_x} = -\frac{e v_x |A|}{c}\sigma_x && (1.7)\\
H'_y &= -\frac{e|A|}{\hbar c}\frac{\partial H_0}{\partial k_y} = -\frac{e v_y |A|}{c}\sigma_y && (1.8)
\end{aligned}$$

So Fermi's Golden rule gives (for both dir $= x, y$):

$$w_{\text{dir}} = \frac{2\pi}{\hbar}\left(\frac{e v_{\text{dir}}|A|}{c}\right)^2 \frac{2\sin^2(\theta) + 2\cos^2(\theta)}{4} D(E_f) = \frac{\pi}{\hbar}\left(\frac{e v_{\text{dir}}|A|}{c}\right)^2 D(E_f) = \langle w_{\text{dir}} \rangle \tag{1.9}$$

The number of states per volume is (with $E = \hbar\sqrt{(v_x k_x)^2 + (v_y k_y)^2}$) (recall the area of ellipse is $\pi r_1 r_2$):

$$\frac{N}{V} = \frac{\pi k_x^{\max} k_y^{\max}}{(2\pi)^2} = \frac{\pi E^2}{(2\pi)^2 \hbar v_x v_y} \tag{1.10}$$

So the density of states is (with $E_f = \hbar\omega/2$):

$$D(E_f) = \frac{\hbar\omega}{4\pi\hbar^2 v_x v_y} \tag{1.11}$$

So, we see that the absorption is:

$$\begin{aligned}
P_{\text{dir}}(\omega) &= \frac{\langle w \rangle \hbar\omega}{W_{\text{incident}}} && (1.12)\\
&= \hbar\omega\frac{\pi}{\hbar}\left(\frac{e v_{\text{dir}}|A|}{c}\right)^2 \frac{\hbar\omega}{4\pi\hbar^2 v_x v_y}\frac{4\pi c}{\omega^2|A|^2} && (1.13)\\
&= \frac{\pi e^2}{\hbar c}\frac{v_{\text{dir}}^2}{v_x v_y} && (1.14)\\
&= \pi\alpha\frac{v_{\text{dir}}^2}{v_x v_y} && (1.15)
\end{aligned}$$

Which is $\pi\alpha$ when $v_x = v_y$.