Derivation of Position for Driven and Damped Oscillator

Consider a spring-block system with a block of mass m, spring constant k, and drag coefficient b, driven by $F_{\text{drive}} = F_d \sin(\omega t)$, where $x(0) = x_0$ and $v(0) = v_0$. What is $x(t)$?

We recall Newton's Law:

$$\sum_i F_i = ma$$

Now, this is:

$$-kx - bv + F_d \sin(\omega t) = ma$$

Equivalently:

$$m\ddot{x} + b\dot{x} + kx = F_d \sin(\omega t)$$

Now, from the theory of differential equations, the solution of a non-homogeneous linear differential equation is the sum of the solution to the homogeneous differential equation (the steady state behavior), and a solution to the "particular" non-homogeneous equation (the transient behavior):

$$x(t) = x_h(t) + x_p(t)$$

The homogeneous equation may be solved using the method of characteristic equations. The roots are:

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

So we have:

$$x_h(t) = \alpha_+ e^{\lambda_+ t} + \alpha_- e^{\lambda_- t}$$

Now, for the particular solution, we note $\sin(\omega t) = (e^{+i\omega t} - e^{-i\omega t})/2i$, so we assume a solution of the form:

$$x_p(t) = A_+ e^{+i\omega t} + A_- e^{-i\omega t}$$

Where:

$$\begin{aligned} \dot{x}_p(t) &= A_+ i\omega e^{+i\omega t} - A_- i\omega e^{-i\omega t} \\ \ddot{x}_p(t) &= A_+ i^2\omega^2 e^{+i\omega t} + A_- i^2\omega^2 e^{-i\omega t} \end{aligned}$$

So, with $i^2 = -1$:

$$m[-A_+\omega^2 e^{+i\omega t} - A_-\omega^2 e^{-i\omega t}] \ + \ b[A_+ i\omega e^{+i\omega t} - A_- i\omega e^{-i\omega t}] \ + \ k[A_+ e^{+i\omega t} + A_- e^{-i\omega t}] \ = \ F_d \frac{e^{+i\omega t} - e^{-i\omega t}}{2i}$$

Thus we have a system of four equations and four unknowns (boundary values and equating terms):

$$\begin{aligned} x_0 &= \alpha_+ + \alpha_- + A_+ + A_- \\ v_0 &= \alpha_+\lambda_+ + \alpha_-\lambda_- + A_+ i\omega - A_- i\omega \\ \frac{F_d}{2i} &= m[-A_+\omega^2] + b[A_+ i\omega] + k[A_+] \\ -\frac{F_d}{2i} &= m[-A_-\omega^2] + b[-A_- i\omega] + k[A_-] \end{aligned}$$

Whose solutions are:

$$A_\pm = \pm \frac{iF_d}{2(m\omega^2 \mp bi\omega - k)}$$

$$\alpha_\pm = \pm \frac{v_0 + (A_+ + A_- - x_0)\lambda_\mp + (A_- - A_+)i\omega}{\lambda_+ - \lambda_-}$$

Which is a long expression, however the qualitative behavior is explained online and in many textbooks.