Cyclotron motion with a tilt

Let $\vec{v}(t=0) = v\hat{x}$ be the initial velocity of a particle with charge q and mass m, and let the particle be subject to a magnetic field $\vec{B} = B\cos(\theta)\hat{x} + B\sin(\theta)\hat{z}$. Describe the subsequent motion $\vec{x}(t)$ if $\vec{x}(t=0) = (0,0,0)^\top$.

Now, let the parallel and perpendicular components of the velocity be:

$$\vec{v}_\parallel(t=0) = (\vec{v}\cdot\hat{B})\hat{B} = (v\cos^2(\theta), 0, v\cos(\theta)\sin(\theta))^\top \tag{1}$$

$$\vec{v}_\perp(t=0) = \vec{v} - \vec{v}_\parallel = \vec{v} - (\vec{v}\cdot\hat{B})\hat{B} = (v(1-\cos^2(\theta)), 0, -v\cos(\theta)\sin(\theta))^\top \tag{2}$$

The force on the particle are $\vec{F} = \vec{F}_\parallel + \vec{F}_\perp$, where:

$$\vec{F}_\parallel = q\vec{v}_\parallel \times \vec{B} = m\frac{d\vec{v}_\parallel}{dt} \tag{3}$$

$$\vec{F}_\perp = q\vec{v}_\perp \times \vec{B} = m\frac{d\vec{v}_\perp}{dt} \tag{4}$$

Evidently $\vec{v}_\parallel \times \vec{B} = 0$ since these vectors are in the same direction, so we have:

$$\frac{d\vec{v}_\parallel}{dt} = 0 \tag{5}$$

So:

$$\vec{x}_\parallel = \vec{v}_\parallel(t=0)t + \vec{x}(t=0) \tag{6}$$

$$= (vt\cos^2(\theta), 0, vt\cos(\theta)\sin(\theta))^\top \tag{7}$$

Now, we consider the differential equation for $\vec{v}_\perp$ at $t=0$:

$$\frac{d\vec{v}_\perp}{dt} = \frac{qvB}{m}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1-\cos^2(\theta) & 0 & -\cos(\theta)\sin(\theta) \\ \cos(\theta) & 0 & \sin(\theta) \end{vmatrix} \tag{8}$$

$$= \frac{qvB}{m}(-(1-\cos^2(\theta))\sin(\theta) - (\cos(\theta)\sin(\theta))\cos(\theta))\hat{y} \tag{9}$$

$$= -\frac{qvB}{m}\hat{y} \tag{10}$$

Evidently, $d\vec{v}_\perp/dt$ is always perpendicular to $\vec{v}_\perp$ and has magnitude qvB/m, so we infer from the relation for uniform circular motion, $qvB/m = |\vec{v}_\perp|^2/r$, that the motion is a circle about $\vec{x}(t=0) - (m|\vec{v}_\perp|^2/qvB)\hat{y}$:

$$\vec{x}_\perp = (m|\vec{v}_\perp|^2/qvB)(\cos(\omega t)\hat{x}' + \sin(\omega t)\hat{y}) - (m|\vec{v}_\perp|^2/qvB)\hat{y} \tag{11}$$

where $\hat{x}' = \cos(\theta)\hat{x} - \sin(\theta)\hat{z}$, and $\omega = qB/m$, or, with $|\vec{v}_\perp|^2/v = v\sin^2(\theta)$:

$$\vec{x}_\perp = \frac{mv\sin^2(\theta)}{qB}\begin{pmatrix} \cos((qB/m)t)\cos(\theta) \\ \sin((qB/m)t) - 1 \\ -\cos((qB/m)t)\sin(\theta) \end{pmatrix} \tag{12}$$

Combining, we have:

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp = \begin{pmatrix} vt\cos^2(\theta) \\ 0 \\ vt\cos(\theta)\sin(\theta) \end{pmatrix} + \frac{mv\sin^2(\theta)}{qB}\begin{pmatrix} \cos((qB/m)t)\cos(\theta) \\ \sin((qB/m)t) - 1 \\ -\cos((qB/m)t)\sin(\theta) \end{pmatrix} \tag{13}$$