Solving 2nd Order Linear Homogeneous ODEs with Constant Coefficients

Consider the equation:

$$c_2 \frac{d^2x}{dt^2} + c_1 \frac{dx}{dt} + c_0 x = 0$$

One may then write the characteristic equation:

$$c_2\lambda^2 + c_1\lambda^1 + c_0\lambda^0 = 0$$

$$c_2\lambda^2 + c_1\lambda + c_0 = 0$$

Whose roots are:

$$\lambda_\pm = \frac{-c_1 \pm \sqrt{c_1^2 - 4c_0c_2}}{2c_2}$$

The solution is then for some $\alpha_\pm$:

$$x(t) = \alpha_+ e^{\lambda_+ t} + \alpha_- e^{\lambda_- t}$$

Where $\alpha_\pm$ are determined by the boundary values.

Example: simple harmonic oscillator

A spring block system has a restoring force kx and starts with the block at rest at $x = A$. The block's mass is m. Describe the subsequent motions.

Newton's Law states:

$$F = ma$$

So:

$$-kx = ma$$

Or:

$$m\ddot{x} + 0\dot{x} + kx = 0$$

Thus:

$$m\lambda^2 + 0\lambda + k = 0$$

And:

$$\lambda_\pm = \frac{-0 \pm \sqrt{0 - 4mk}}{2m} = \pm i\sqrt{\frac{k}{m}}$$

Hence:

$$x(t) = \alpha_+ e^{i\sqrt{k/m}\,t} + \alpha_- e^{-i\sqrt{k/m}\,t}$$

The boundary values give a system of equations:

$$x(0) = A = \alpha_+ e^{i\sqrt{k/m}\,0} + \alpha_- e^{-i\sqrt{k/m}\,0} = \alpha_+ + \alpha_-$$

$$v(0) = 0 = i\alpha_+\sqrt{\frac{k}{m}}e^{i\sqrt{k/m}\,0} - i\alpha_-\sqrt{\frac{k}{m}}e^{-i\sqrt{k/m}\,0} = i(\alpha_+ - \alpha_-)\sqrt{\frac{k}{m}}$$

Whose solution is $\alpha_+ = \alpha_- = A/2$, so, with $\cos(x) = (e^{ix} + e^{-ix})/2$:

$$x(t) = \frac{Ae^{i\sqrt{k/m}\,t} + Ae^{-i\sqrt{k/m}\,t}}{2} = A\cos(\sqrt{k/m}\,t)$$