Anderson Localization

Spenser Talkington
UPenn Quantum Condensed Matter Group Meeting

12 January 2023

Outline

- Anderson Localization
  - Tunneling through potentials
  - The self-energy
  - Total localization in 1D
  - Finite size scaling
- Many-Body Localization
  - Integrability
  - Level spacing
  - Thermalization
  - MBL to ETH transition

Quantum Tunneling

- Solution to 1D Schrödinger equation in a region with potential U, electronic energy E

$$\psi(x) = Ae^{-\alpha x} + Be^{\alpha x}$$

$$\alpha(x) = \sqrt{\frac{2m(U - E)}{\hbar}}$$

- $\alpha(x)$ is imaginary for E>U which corresponds to delocalized wave motion
- $\alpha(x)$ is real for U>E, but can still tunnel through any barrier (at long time)
- See also: Klein tunneling, Ramsauer-Townsend effect

Green's Function Propagator

- The retarded Green's function is

$$G(k,E) = \frac{1}{E - H(k) + i\eta}$$

- The dressed propagator is

$$\widetilde{G} = G + G\Sigma_R G$$

where Σ_R is the reducible self-energy

- We then have

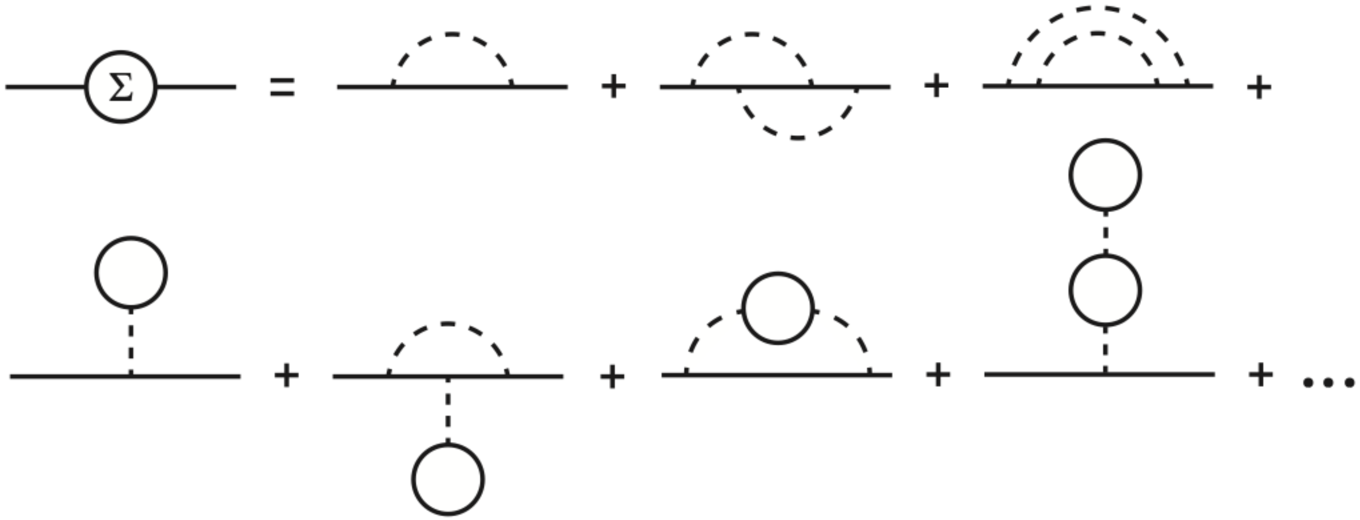
$$\tilde{G}(k,E) = \frac{1}{E - (H(k) + \Sigma) + i\eta}$$

Irreducible Self-Energy

- In terms of the irreducible self energy Σ

$$\begin{aligned}\widetilde{G} &= G + G\Sigma\widetilde{G} \\ &= G + G\Sigma G + G\Sigma G\Sigma\widetilde{G} \\ &= G + G\Sigma G + G\Sigma G\Sigma G + G\Sigma G\Sigma G\Sigma\widetilde{G}\end{aligned}$$

- Where diagrammatically for phonons (to two-loop order)



Anderson Model

- Let ϵ be randomly distributed, and V_{ij} some hopping terms

$$H = \sum_i \epsilon_i \, c_i^+ c_i + \sum_{i<j} V_{ij} c_i^+ c_j$$

Then the time-independent Schrödinger equation gives for eigenstate |a⟩ corresponding to E^n (Abou-Chacra, Thouless, Anderson, JPC **6**, 1734 (1973))

$$\epsilon_i a_i^n + \sum_j V_{ij} a_j^n = E^n a_i^n$$

- The Green's function then satisfies

$$(E - \epsilon_i)\tilde{G}_{ik}(E) + \sum_j V_{ij} \tilde{G}_{jk}(E) = \delta_{ik}$$

- We then have the site self energy Σ_i which is *defined* by

$$E - \epsilon_i - \Sigma_i(E) = \frac{1}{\tilde{G}_{ii}(E)}$$

Localization

- We have (letting E and η be strictly real)

$$\tilde{G}_{ii}(E) = \frac{1}{E + i\eta - (\epsilon_i + \Sigma_i(E + i\eta))}$$

- Taking the limit $\eta \to 0$

$$\tilde{G}_{ii}(E) = \lim_{\eta \to 0} \frac{1}{E - (\epsilon_i + \Sigma_i(E + i\eta))}$$

For localized states $\lim_{\eta \to 0} \mathrm{Im}[\Sigma_i(E + i\eta))] = 0$

For delocalized states $\lim_{\eta \to 0} \mathrm{Im}[\Sigma_i(E + i\eta))] \neq 0$

Self-Consistency for the Self-Energy

- Let the site-excluded Green's function be (with *jk*, … excluded)

$$\tilde{G}_i^{(jk\dots)}(E) = \frac{1}{E - \epsilon_i - \Sigma_{\mathrm{i}}^{(jk)}(E)}$$

- The site self-energy is then (Feenburg, Phys. Rev. **74**, 206 (1948))

$$\Sigma_i(E) = \sum_{j\neq i} V_{ij}\tilde{G}_j^{(i)}V_{ji} - \sum_{j\neq i}\sum_{k\neq i,j} V_{ik}\tilde{G}_k^{(ij)}V_{kj}\tilde{G}_j^{(i)}V_{ji} + \cdots$$

- Anderson then showed (Anderson, Phys. Rev. **109**, 1492 (1958)) that when the series is convergent Σ_i has zero imaginary part (since all low-order terms have zero imaginary part), and corresponds to localized states. If the series is divergent then Σ_i can have non-zero imaginary part and correspond to delocalized states.

Physical Picture of the Self-Consistency

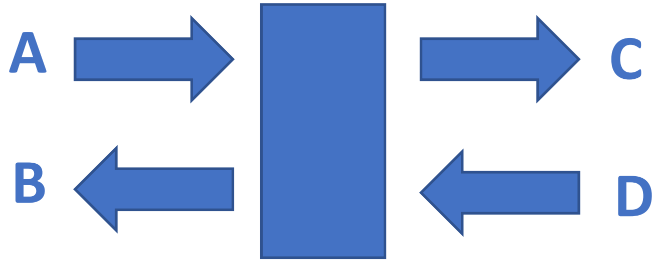
- We have

$$\Sigma_i(E) = \sum_{j\neq i} V_{ij}\tilde{G}_j^{(i)}V_{ji} - \sum_{j\neq i}\sum_{k\neq i,j} V_{ik}\tilde{G}_k^{(ij)}V_{kj}\tilde{G}_j^{(i)}V_{ji} + \cdots$$

- Now, this is a self-avoiding walk on a closed loop!
- The condition for convergence is then that of destructive interference so that phases cancel, and propagation is prevented (imagine counter-propagating walks: the phases then cancel)
- Note: This analysis only holds when V are purely real, if they are imaginary, as in for example in the presence of a B field, states can be delocalized—this delocalization corresponds to the edge states

Total Localization in 1D (physical sketch)

- Determine the transmission and reflection coefficients of a wavepacket through one site
  - Using the scattering matrix
    - $S = \begin{pmatrix} r & t^* \\ t & r^* \end{pmatrix}$
    - $S\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix}$



- Find the resistance $\rho_i = r_i r_i^* / t_i t_i^*$ → $\langle \rho \rangle = \frac{1}{N} \sum_i \rho_i$
- Take the average, as $N \to \infty$ (for nearest neighbor hopping t, see reference)

$$\langle \rho \rangle \propto \left(\frac{\langle \epsilon \rangle^2}{2t^2} + \sqrt{1 + \frac{\langle \epsilon \rangle^2}{2t^2}} \right)^N$$

- Alternatively, there is *always* destructive interference (a walk can't self avoid, and its reverse path is itself) so the series for the self-energy converges and states are localized

Finite-Size Scaling Theory

- Finite size scaling hypothesis (FSS)—“homogeneous functions”
$$g(bL) = g(b)f(L)$$
- If FSS holds for all L then if we know g at one length scale, we know it all length scales! Can find critical points, and differentiate to find critical exponents

- Fun result: total localization in 2D
  - Abrahams, Anderson, Licciardello, and Ramakrishnan showed total localization in 2D.
  - Only when hopping is purely real!

Thouless Number/Energy and Level Spacing

- The scaling function g we are interested in for localization is the "Thouless number" which is a dimensionless conductance

$$g(E,L) = \Delta E\, D(E,L)$$

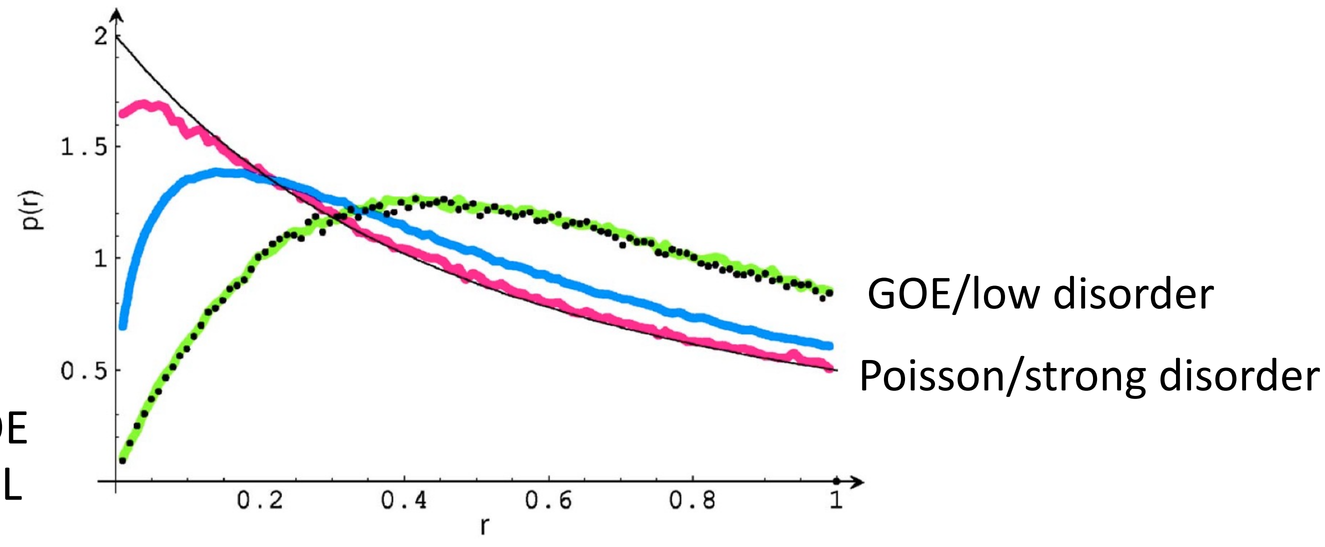
by $D(E) = dN/dE(L)$, we mean 1/mean spacing on the energy levels, which for sufficiently large dE becomes the density of states. ΔE is the change in energy as periodic boundary conditions are replaced with anti-periodic boundary conditions (2π vs 4π periodicity). (See Thouless, Physics Reports **13**, 93 (1974))

- Small energy level spacing leads to increased conductance
- Recall Fermi's golden rule

$$\text{Transition rate} = \frac{2\pi}{\hbar} |\langle F|H'|I\rangle|^2 D(E_F)$$

Random-Matrix Theory and Level Spacing

- Three symmetry classes for random matrices
  - GOE: TRS preserving
  - GUE: TRS broken
  - SE: TRS preserving spin-1/2
- PRB **47**, 11487 (1993)
  - Localized: Poisson (no hybridization)
  - Delocalized (ergodic/thermalized/quantum chaotic/non-integrable/few locally conserved quantities/hybridizes to level repulsion): GOE/GUE
- PRB **75**, 155111 (2007)
  - See the crossover between Poisson and GOE at finite temperature with interactions: MBL
  - Much subsequent woek



Anderson Localization with Interactions

- With interactions and temperature we can no longer consider just the ground state, but need to consider properties of excited states too
- Fleishman and Anderson, PRB **21**, 2366 (1980)
  - For sufficiently weak and short range disorder, localization seems to hold
- Is there Anderson localization at finite temperatures/with interactions?
  - At long times information about the initial state is typically lost to thermalization (maximally entangled state—density matrix proportional to 1)
  - Anderson localization would provide a way out (other than finely tuned integrable systems)
  - Proposed “many-body localization”

Is it MBL, Quantum Chaos, or a Glass?

- Analytic arguments, and small-scale numeric studies in support of MBL
- Random-matrix results are only true for infinite matrices
- Recent soul-searching in the MBL community over how trustworthy their analyses are
  - PRE **102**, 062144 (2020) vs Ann. Phys. **427**, 168415 (2021)
  - PRE **104**, 054105 (2021), PRB **104**, L201117 (2021)

Summary: Perspectives on Localization

- Standing or traveling wave
  - Does self energy have an imaginary part?
- Scaling for the conductance
  - Which way does the RG flow go?
- Level spacing statistics
  - Is it Wigner-Dyson or Poisson?
- Entanglement entropy
  - Is it extensive or sub-extensive?