VI. A: General Formalism

Planar spins can be written as

$$s(\vec{n}) = \begin{pmatrix} \cos(\theta(\vec{n})) \\ \sin(\theta(\vec{n})) \end{pmatrix}$$

where $\vec{n}$ indicates the position in space (on the lattice).
A action is then

$$S = -J \sum_{\vec{n},\vec{\mu}} s(\vec{n}) \cdot s(\vec{n}+\vec{\mu})$$

which with

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

becomes

$$S = -J \sum_{\vec{n},\vec{\mu}} \cos(\theta(\vec{n}) - \theta(\vec{n}+\vec{\mu}))$$

where observables will be derived from the generating functional

$$Z = e^{iS/\hbar}$$

One can then introduce the difference operator

$$\Delta_{\vec{\mu}}\theta(\vec{n}) = \theta(\vec{n}+\vec{\mu}) - \theta(\vec{n})$$

so that

$$S = -J \sum_{\vec{n},\vec{\mu}} \cos(\Delta_{\vec{\mu}}\theta(\vec{n}))$$

Note that

$$\theta(\vec{n}+\vec{\mu}) - \alpha - (\theta(\vec{n}) - \alpha) = \theta(\vec{n}+\vec{\mu}) - \theta(\vec{n})$$

so the spins can be rotated through an angle α without changing the action. I.e. $\theta(\vec{n}) \mapsto \theta(\vec{n}) - \alpha$.

Now let us consider a different model for which there are spins on the *links* between sites $(\vec{n},\vec{\mu})$. Clearly $(\vec{n},\vec{\mu}) = (\vec{n}+\vec{\mu},-\vec{\mu})$ so the angular variable $\theta_{\vec{\mu}}(\vec{n})$ on $(\vec{n},\vec{\mu})$ should be related to the angular variable on $(\vec{n}+\vec{\mu},-\vec{\mu})$. Here we choose to tilt our heads so that

$$\theta_{-\vec{\mu}}(\vec{n}+\vec{\mu}) = -\theta_{\vec{\mu}}(\vec{n})$$

One can then define the discrete curl

$$\theta_{\vec{\mu}\vec{\nu}} = \Delta_{\vec{\mu}}\theta_{\vec{\nu}}(\vec{n}) - \Delta_{\vec{\nu}}(\vec{n})\theta_{\vec{\mu}}(\vec{n})$$

which is an antisymmetric tensor just like $\epsilon_{\mu\nu}$. Now, by the definition of $\Delta_{\vec{\mu}}$ we have

$$\theta_{\vec{\mu}\vec{\nu}} = \theta_{\vec{\nu}}(\vec{n}+\vec{\nu}) - \theta_{\vec{\nu}}(\vec{n}) - \theta_{\vec{\mu}}(\vec{n}+\vec{\mu}) + \theta_{\vec{\mu}}(\vec{n})$$

or shuffling terms and using $\theta_{-\vec{\mu}}(\vec{n}+\vec{\mu}) = -\theta_{\vec{\mu}}(\vec{n})$ this becomes

$$\theta_{\vec{\mu}\vec{\nu}} = \theta_{\vec{\mu}}(\vec{n}) + \theta_{\vec{\nu}}(\vec{n}+\vec{\mu}) + \theta_{-\vec{\mu}}(\vec{n}+\vec{\mu}+\vec{\nu}) + \theta_{-\vec{\nu}}(\vec{n}+\vec{\nu})$$

As before we have invariance of this curl since under the gauge transformation

$$\theta_{\vec{\mu}} \mapsto \theta_{\vec{\mu}}(\vec{n}+\vec{\mu}) - \chi(\vec{n})$$

since

$$\theta_{-\vec{\mu}}(\vec{n}+\vec{\mu}) \mapsto \theta_{-\vec{\mu}}(\vec{n}+\vec{\mu}) + \chi(\vec{n})$$

so for each positive direction θ there is a negative direction θ and the positive and negative χ terms cancel.

Now let us recall that in electromagnetism there is an antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

that is gauge invariant under gauge transformations

$$A_\mu \mapsto A_\mu + \partial_\mu \chi$$

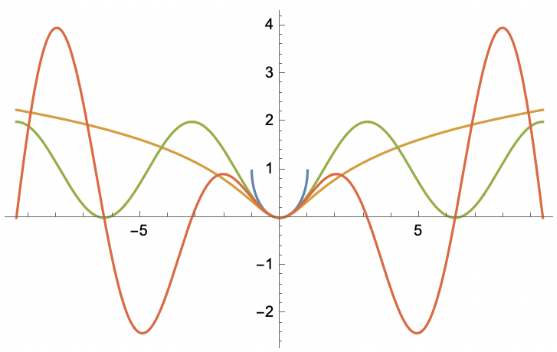
The Maxwell action is then

$$S_M = \frac{1}{4}\int d^D x\ F_{\mu\nu}F_{\mu\nu}$$

The discrete curl is a rank-2 antisymmetric tensor so we may be able to contrive an action for our spin model that recreates electromagnetism to lowest order in $\theta_{\vec{\mu}\vec{\nu}}$. Four such actions are

$$S_1 = J\sum_{\vec{n},\vec{\mu},\vec{\nu}}(1-\sqrt{1-\theta^2_{\vec{\mu}\vec{\nu}}(\vec{n})})$$
$$S_2 = \frac{J}{2}\sum_{\vec{n},\vec{\mu},\vec{\nu}}\ln(1+\theta^2_{\vec{\mu}\vec{\nu}}(\vec{n}))$$
$$S_3 = J\sum_{\vec{n},\vec{\mu},\vec{\nu}}(1-\cos(\theta_{\vec{\mu}\vec{\nu}}(\vec{n})))$$
$$S_4 = \frac{J}{2}\sum_{\vec{n},\vec{\mu},\vec{\nu}}\sin(\theta_{\vec{\mu}\vec{\nu}}(\vec{n}))\theta_{\vec{\mu}\vec{\nu}}(\vec{n})$$

Let's think about these actions.



The first action isn't real for $|\theta_{\vec{\mu}\vec{\nu}}| > 1$ so we throw it out, even though we will be interested in physics for $\theta_{\vec{\mu}\vec{\nu}} \ll 1$. The second action has a global minimum at $\theta_{\vec{\mu}\vec{\nu}} = 0$ and monotonically increases with $|\theta_{\vec{\mu}\vec{\nu}}|$. The third action is periodic. The fourth action oscillates, but the minimum at zero isn't the global minimum so we disregard it. So we are left with the second and third actions. Now Kogut chooses the third one since it is periodic in $\theta_{\vec{\mu}\vec{\nu}}$ so that it is something we can plausibly believe recreates the physics of a lattice—even though by expanding for small $\theta_{\vec{\mu}\vec{\nu}}$ we inherently don't care about the periodicity of the action. Here we refrain from making a choice.

Let us now expand so that

$$S = \frac{J}{2}\sum_{\vec{n},\vec{\mu},\vec{\nu}}\theta^2_{\vec{\mu}\vec{\nu}}(\vec{n})$$

If we squint at this it becomes

$$S = \frac{J}{2}\int \frac{d^D x}{a^D}\theta_{\vec{\mu}\vec{\nu}}(\vec{x})\theta_{\vec{\mu}\vec{\nu}}(\vec{x})$$

where we use implicit summation over the $\vec{\mu}$ and $\vec{\nu}$ indices.

We can then insist that these really are the same theory and compare terms and see that

$$\left(\frac{J}{2a^D}\right)^{1/2} \theta_{\vec{\mu}\vec{\nu}}(\vec{x}) = \frac{1}{2}F_{\mu\nu}$$

or, identifying μ with $\vec{\mu}$ and ν with $\vec{\nu}$, we have

$$A_\mu(\vec{x}) = \frac{1}{a}\underbrace{\left(\frac{a^D}{2J}\right)^{1/2}}_{\text{“}1/g\text{”}} \theta_{\vec{\mu}}(\vec{x})$$

where g is the coupling constant of the theory, and the $1/a$ comes from the discrete going to continuous derivative.

One can then make deep philosophical points like: “oh my, is the universe really made of microscopic degrees of freedom whose low energy behavior gives electromagnetism.” Sure, but *which* degrees of freedom? As we saw there are an infinite number of actions that lead to the same low energy behavior.

Aside About Wilson Loops

Now we have a coupling constant between a charge (recall electromagnetism supports quantization of a massless spin-1 gauge boson). One can then draw this diagrammatically. One might imagine that the rule is

$$\bar{u}(\vec{x}+\vec{\xi})\gamma_\mu u(\vec{x})$$

for Dirac matrices γ and translations $\vec{\xi}$ in real space. But this isn't gauge invariant under $A_\mu(\vec{x}) \mapsto A_\mu(\vec{x}) + \partial_\mu \Lambda(\vec{x})$ since we have

$$\bar{u}(\vec{x}+\vec{\xi})\gamma_\mu u(\vec{x}) \mapsto e^{-ie\Lambda(\vec{x}+\vec{\xi})}\bar{u}(\vec{x}+\vec{\xi})\gamma_\mu e^{ie\Lambda(\vec{x})}u(\vec{x})$$

where the exponentials only cancel if $\Lambda(\vec{x}) = \Lambda(\vec{x}+\vec{\xi})$.

We can squint at this and say “a ha, I know the fundamental theorem of calculus so a gauge invariant quantity is”

$$\bar{u}(\vec{x}+\vec{\xi})\gamma_\mu e^{ie\int_{\vec{x}}^{\vec{x}+\vec{\xi}} dx^\mu A_\mu} u(\vec{x})$$

since

$$A(x+\xi) = A(x) + \int_x^{x+\xi} d\bar{x}\ A'(\bar{x})$$

This is known as the Peierls substitution, which is also attributed to Schwinger. This is also known as the Wilson line

$$W(\vec{x}, \vec{x}+\vec{\xi}) = e^{ie\int_{\vec{x}}^{\vec{x}+\vec{\xi}} dx^\mu A_\mu}$$

If we close a Wilson line so that it traverses a contour C then

$$W(C) = e^{ie\oint_C dx^\mu A_\mu}$$

which is known as the Wilson loop for contour C over connection A. It is straightforward to show that a Wilson loop is independent of basepoint $\vec{x}$. (Note that this exponential is path-ordered which is important for non-Abelian connections).
In the language of differential geometry, we refer to this as a holonomy over a fiber bundle, where A is the connection that defines the parallel transport.

VI. C: The Quantum Hamiltonian Formulation

In this section we will stop using vector notation, i.e. $n = \vec{n}$.
Let us consider an action with anisotropic couplings (note the suggestive naming β)

$$S = 2\beta_\tau \sum_{n,j} (1 - \cos(\theta_{0j})) - \beta_x \sum_{n,i,j} \cos(\theta_{ij}(n))$$

where we choose a gauge so that

$$\theta_0(n) = 0$$

from which it follows that τ-independent gauge transformations are *local* symmetries of the system.
We then have for β_τ infinite, infinitesimal θ_{0j} so that

$$1 - \cos(\theta_{0j}) \approx \frac{1}{2}\theta_{0j}^2 = \frac{1}{2}a_\tau^2 \left(\frac{\partial \theta_j}{\partial \tau}\right)^2$$

In the infinitesimal spacing limit we can also take

$$\sum_{n,j} \to \int \frac{d\tau}{a_\tau} \sum_{n,j}$$

where the n on the left is $n = (n_\tau, n_x)$ while the right is $n = (0, n_x)$.
The action is then

$$S = \int d\tau \; \left(\beta_\tau a_\tau \sum_{n,j} (\partial_\tau \theta_j(\tau, n))^2 - \frac{\beta_x}{a_\tau} \sum_{n,i,j} \cos(\theta_{ij}(\tau, n)) \right)$$

One can then look at this in terms of the small a_τ limit. In this limit we see that $\beta_\tau \sim 1/a_\tau$ and $\beta_x \sim a_\tau$ must hold for the behavior to be well defined. Let

$$\beta_\tau = g_\tau^2 / a_\tau$$
$$\beta_x = g_x^2 a_\tau$$

where g_τ and g_x are real scalars. Substituting, we have

$$S = \int d\tau \; \left(g_\tau^2 \sum_{n,j} (\partial_\tau \theta_j(\tau, n))^2 - g_x^2 \sum_{n,i,j} \cos(\theta_{ij}(\tau, n)) \right)$$

One can then rescale $g_\tau^2 = sg^2$ and $g_x^2 = s/g^2$ so that

$$S = s \int d\tau \; \left(g^2 \sum_{n,j} (\partial_\tau \theta_j(\tau, n))^2 - \frac{1}{g^2} \sum_{n,i,j} \cos(\theta_{ij}(\tau, n)) \right)$$

where s sets the overall scale of the action. Now if we look at this we see that it is ripe for the application of perturbation theory. If $g \gg 1$ then the second term can be treated as a perturbation, while if $g \ll 1$ then the the first term can be treated as a pertubration.

Now, we want to deal with a Hamiltonian and not a Lagrangian density/action, so we consider the conjugate momentum to $\theta_j(n)$—call it $L_j(n)$. The position and its conjugate momentum satisfy

$$[x, p] = i$$

so it seems reasonable to believe that

$$[\theta_i(n), L_j(m)] = i\,\delta_{ij}\delta_{nm}$$

where we differ from Kogut by rotating the spins by π so that $\theta_i(n) \mapsto \theta_i(n) + \pi = -\theta_i(n)$ so that the structure of the commutator is the same as the one we are familiar with. Now, this commutation relation is expected based on the general relation between position and conjugate momentum, and also based on the relation between periodicity in space under rotations for usual angular momentum, and periodicity in space under translations for this analogue to angular momentum.

To go from a Lagrangian density to a Hamiltonian density we have

$$\begin{aligned} H &= p\dot{x} - L \\ &= (\sum_j L_j\dot{\theta}_j) - L \end{aligned}$$

So taking this similar procedure here we have the Hamiltonian

$$a_x H = \sum_{n,j} g^2 L_j^2(n) - \frac{1}{g^2}\sum_{n,i,j}\cos(\theta_{ij}(n))$$

Now, τ-independent local gauge transformations will add the same angle to all the spins connected to a vertex. An operator that does this for us is

$$G_\chi(n) = e^{-i\sum_{\pm j} L_j(n)\chi}$$

which is the rotation operator that takes

$$\theta_i(n) \mapsto \theta_i(n) - \chi$$

for all links i connected to vertex n. So a local gauge transformation that acts on all sites in the lattice is

$$G(\chi) = e^{-i\sum_{n,\pm j} L_j(n)\chi(n)}$$

for which the Hamiltonian is gauge invariant since to first order in the exponential

$$G(\chi)\theta_j(n)G^{-1}(\chi) = \theta_j(n) - \chi(n) + \chi(n+j) = \theta_j(n) + \Delta_j\chi$$

so that

$$G(\chi)HG^{-1}(\chi) = H$$

where G is a unitary matrix that squares to the identity and commutes with the Hamiltonian

$$[G(\chi), H] = 0$$

Also, Elitzur's theorem ensures that the space of states is invariant under this gauge transformation too.

Now, we consider the commutation relation

$$[\theta_i(n), L_j(m)] = i\delta_{ij}\delta_{nm}$$

From Section VI. A, we have

$$A_\mu(n) = \frac{1}{a_\mu} \underbrace{\left(\frac{a_x^D}{2J}\right)^{1/2}}_{\text{“}1/g\text{”}} \theta_\mu(n)$$

for positions n and coupling constant g. This means that we can rewrite

$$a_i g[A_i(n), L_j(m)] = i\delta_{ij}\delta_{nm}$$

or, with $\lim_{a\to 0} \delta_{nm}/a_x^3 = \delta(r_n - r_m)$ where for the continuum limit we write r_n for n and r_m for m. We then have

$$\frac{a_i g}{a_x^3}[A_i(n), L_j(m)] = i\delta_{ij}\delta(r_n - r_m)$$

and we can name $E_j(m) = (a_j g/a_x^3)L_j(m)$ so that

$$[A_i(n), E_j(m)] = i\delta_{ij}\delta(r_n - r_m)$$

We can now say a few things. Firstly, by the identications above we have pushed the lattice commutation relation into the commutation relation for the continuum theory of electromagnetism where A is the vector potential and E is the scalar potential. Additionally since θ was an angular position (which is periodic) and L was its conjugate momentum, L has a discrete spectrum given by $L_j(n) \in \mathbb{Z}$. Combining these, we see that the electric field arising from this lattice theory is discretized: “Electric flux cannot subdivide into arbitrarily small units on individual links,” where the quantized charge is proportional to g. This originates from the fact that A lives on a compact manifold—here S^1, i.e. the underlying gauge group is $U(1)$; if we had chosen a different underlying gauge group we could have a different notion of quantized charge (and we could get a non-Abelian gauge theory).

Having made these identifications, we can return to the Hamiltonian

$$a_x H = \sum_{n,j} g^2 L_j^2(n) - \frac{1}{g^2}\sum_{n,i,j}\cos(\theta_{ij}(n)) = \sum_{n,j}\frac{a_x^6}{a_j^2}E_j^2(n) - \frac{1}{g^2}\sum_{n,i,j}\cos(\theta_{ij}(n))$$

and we see that the first term looks a lot like the energy stored in the electric field

$$\int_{\text{vol}} d^{D-1}x\ E^\mu E_\mu$$

We then hope that the second term will be like the energy stored in the magnetic field

$$\int_{\text{vol}} d^{D-1}x\ B^\mu B_\mu$$

So with $\cos(\theta_{ij}(n)) \approx 1 - \theta_{ij}^2(n)/2$ we identify

$$\theta_{ij} = \sqrt{2}a_x^2 g B_k$$

so that

$$a_x H = \sum_{n,j}\frac{a_x^6}{a_k^2}E_j^2(n) + \sum_{n,k} a_x^4 B_k^2(n)$$

whence with $a_j = a_x$

$$H = \sum_{n,j} a_x^3 E_j^2(n) + a_x^3 B_j^2(n)$$

where we arrive at our Hamiltonian (for $D = 3+1$).

Now that we have seen that a lattice model can recreate key features of classical electromagnetism, let us recreate Gauss's law. We generalize these conjugate momenta L so that the generator of local gauge transformations is

$$L(n) + \sum_{j \in \mathbb{Z}} L_j(n)$$

so that

$$G(\chi) = e^{-i \sum_n L(n)\chi(n) - i \sum_{n,j} L_j(n)\chi(n)}$$

whence

$$G(\chi) e^{\pm i\theta(n)} G^{-1}(\chi) = e^{\pm i(\theta(n) - \chi(n))}$$

Now we seek an operator that places charges $\pm g$ at positions 0 and R respectively. A plausible gauge-invariant quantity is

$$\theta_C(0, R) = e^{i\theta(0)} e^{-i \sum_{i \in C} \theta_{C_i}(n)} e^{-i\theta(R)}$$

where $e^{-i \sum_C \theta_{C_i}(n)}$ is the connection.

Due to the commutation relation, this leads to the a unit of flux being generated on each link along C, which is precisely Gauss's law.

Now we ask: which contour minimizes the energy?

If $g^2 \gg 1$ then the "electric" term dominates and the energy per link is a positive constant so the minimum energy contour is the shortest contour, and the energy of the contour grows linearly in R. This is known as "quark confinement" since the coefficient is large, and corrections to this make it favorable to generate new quarks ($E = mc^2$) rather than to increase the length of this contour, or "flux tube".

If $g^2 \ll 1$, e.g. if $g^2 = 4\pi/137.036$, then normal electromagnetism is recovered (where energy does not grow with path length and charges are not confined) and the contours are those of the electric dipole which permeate space.