Non-Abelian Fundamental Groups, the Second Homotopy Group and Exact Sequences

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University of Pennsylvania
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Outline

- Two (somewhat) disjoint topics

- 1. Non-Abelian fundamental groups
- 2a. The second homotopy group
- 2b. Applications to quantum condensed matter physics

VI. Non-Abelian Fundamental Groups

Biaxial Nematic Liquid Crystals

- Treat as rectangular prisms
  - Identity
  - Inversion
  - π rotations (3)
  - -π rotations (3)
- This is a group
  - The quaternion group
  - It is non-abelian

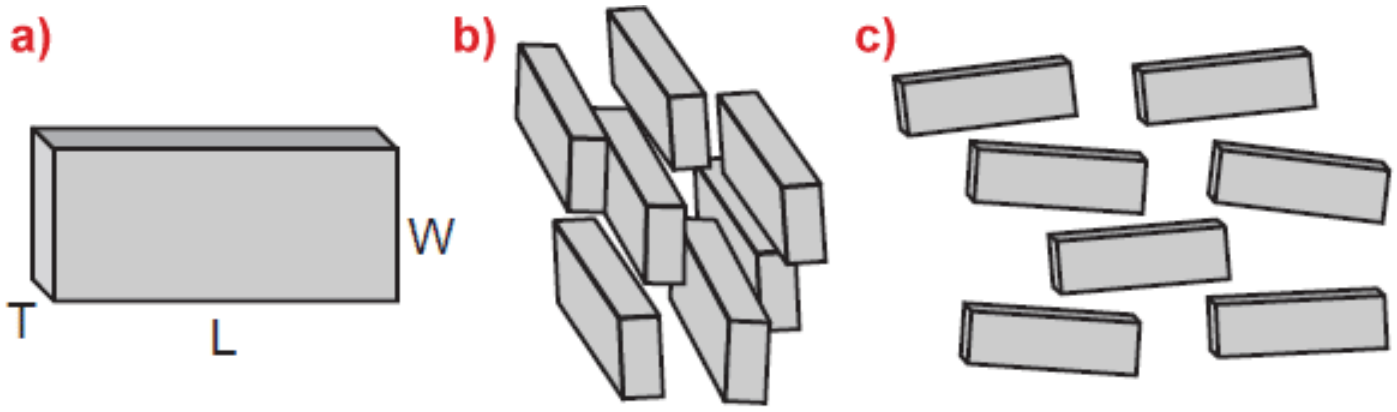


Image Credit: ESRF

Quaternions

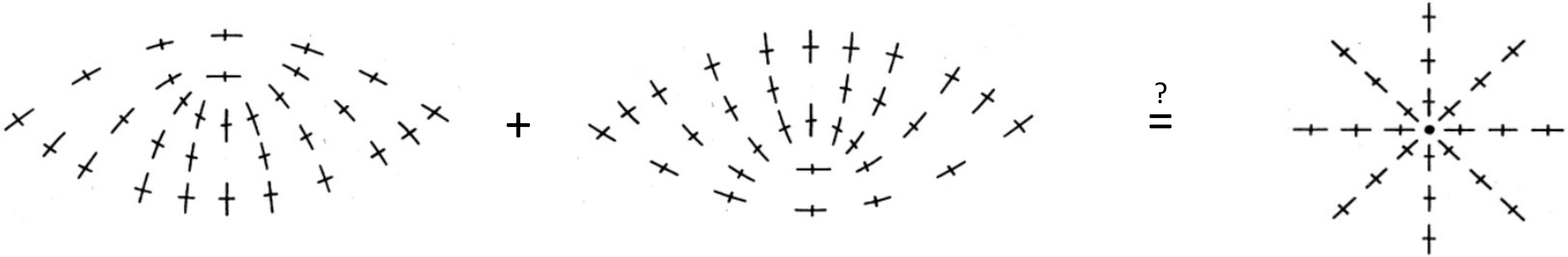
- Representation w/ Pauli matrices
  - H = {±1, i=±is_x, j=±is_y, k=±is_z}
- Reduces to 5 conjugacy classes
  - C_0, $\underline{C}_0$, C_x, C_y, C_z
- Group multiplication table
  - Some elements are non-commuting
- Class multiplication table
  - Class multiplication is abelian so there is no notion of enclosing defects first

	1	i	−1	-i	j	k	−j	−k
1	1	i	−1	-i	j	k	−j	−k
i	i	−1	−i	1	j	−j	−k	j
−1	−1	−i	1	i	−j	−k	j	k
−i	−i	1	i	−1	−k	j	k	−j
j	j	−k	−j	k	−1	i	1	−i
k	k	j	−k	−j	−i	−1	i	1
−j	−j	k	j	−k	1	−i	−1	i
−k	−k	−j	k	j	i	1	−i	−1

	C_0	$\overline{C}_0$	C_x	C_y	C_z
C_0	C_0	$\overline{C}_0$	C_x	C_y	C_z
$\overline{C}_0$	$\overline{C}_0$	C_0	C_x	C_y	C_z
C_x	C_x	C_x	$2C_0+2\overline{C}_0$	$2C_z$	$2C_y$
C_y	C_y	C_y	$2C_z$	$2C_0+2\overline{C}_0$	$2C_x$
C_z	C_z	C_z	$2C_y$	$2C_x$	$2C_0+2\overline{C}_0$

The 2π Point Defect is not the Trivial Defect

- One can imagine two line defects (generated by C_i for example) forming a 2π defect of annihilating each other
- The question: When will two π defects merge to create a 2π defect versus annihilating to a trivial defect?



Ex. Two z-Disclinations

- Case 1: no other disclinations
  - Combines to a 2π point defect

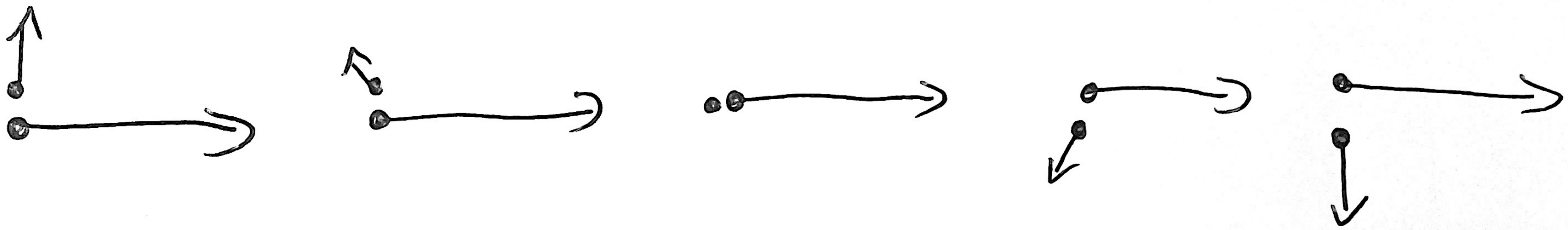
$$-1 = (i\sigma_z)(i\sigma_z)$$

- Case 2: one other disclination
  - Decays via “catalysis”

$$-(i\sigma_z) = (i\sigma_x)(i\sigma_z)(i\sigma_x)^{-1}$$

$$1 = (i\sigma_z)(-i\sigma_z)$$

Schematic of part of the decay process



Overall Guidance

- Based → fundamental group elements
  - The order matters

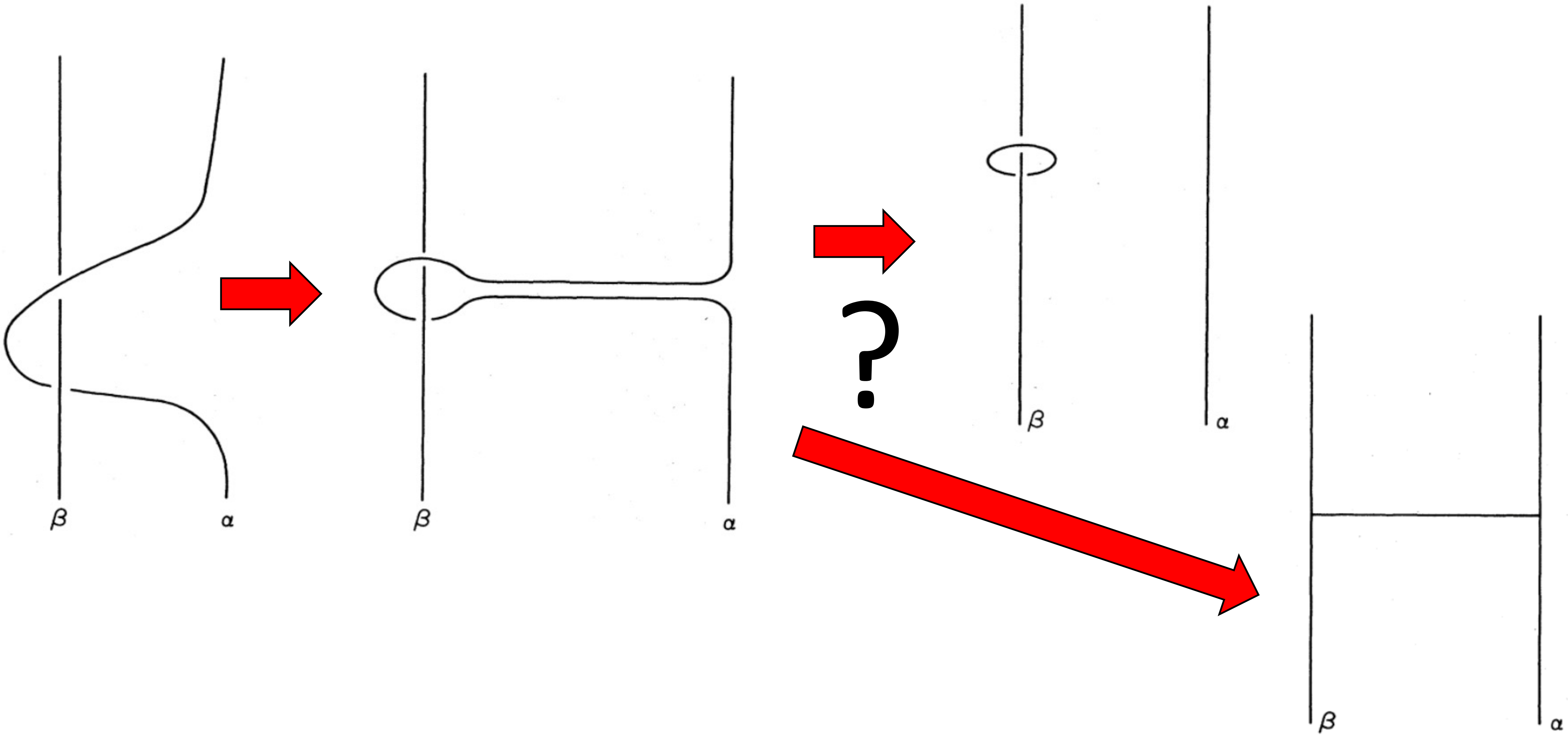
$$(i\sigma_x)(i\sigma_z)(i\sigma_z) \neq (i\sigma_z)(i\sigma_x)(i\sigma_z)$$

- Un-based → conjugacy classes
- Conjugacy classes can be ambiguous
  - This ambiguity is lifted by the contour and what is within it
  - Conjugacy classes combine in an Abelian fashion
  - Think about the example from the last slide

Scarring: the Unique Feature of NA Groups

- For Abelian groups the order line defects cross doesn't matter
- For non-Abelian groups "scarring" can occur
  - An effect where the crossing of line defects generates another line defect
- Scarring is only avoided when $\beta\alpha\beta^{-1}\alpha^{-1} = 1$
  - But this only holds for commuting elements α and β
- Let's look at this pictorially

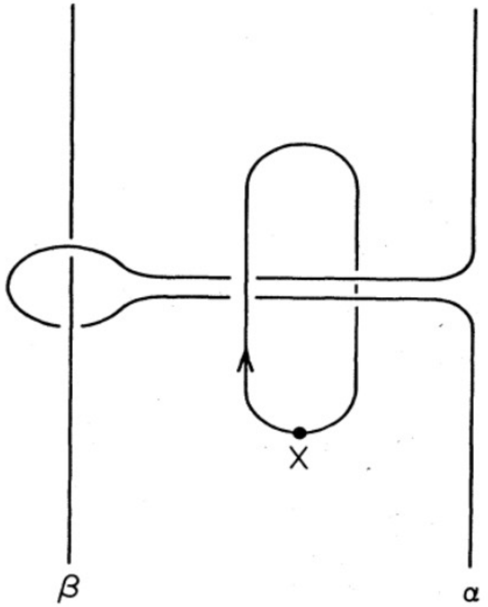
To Scar or not to Scar, that is the Question

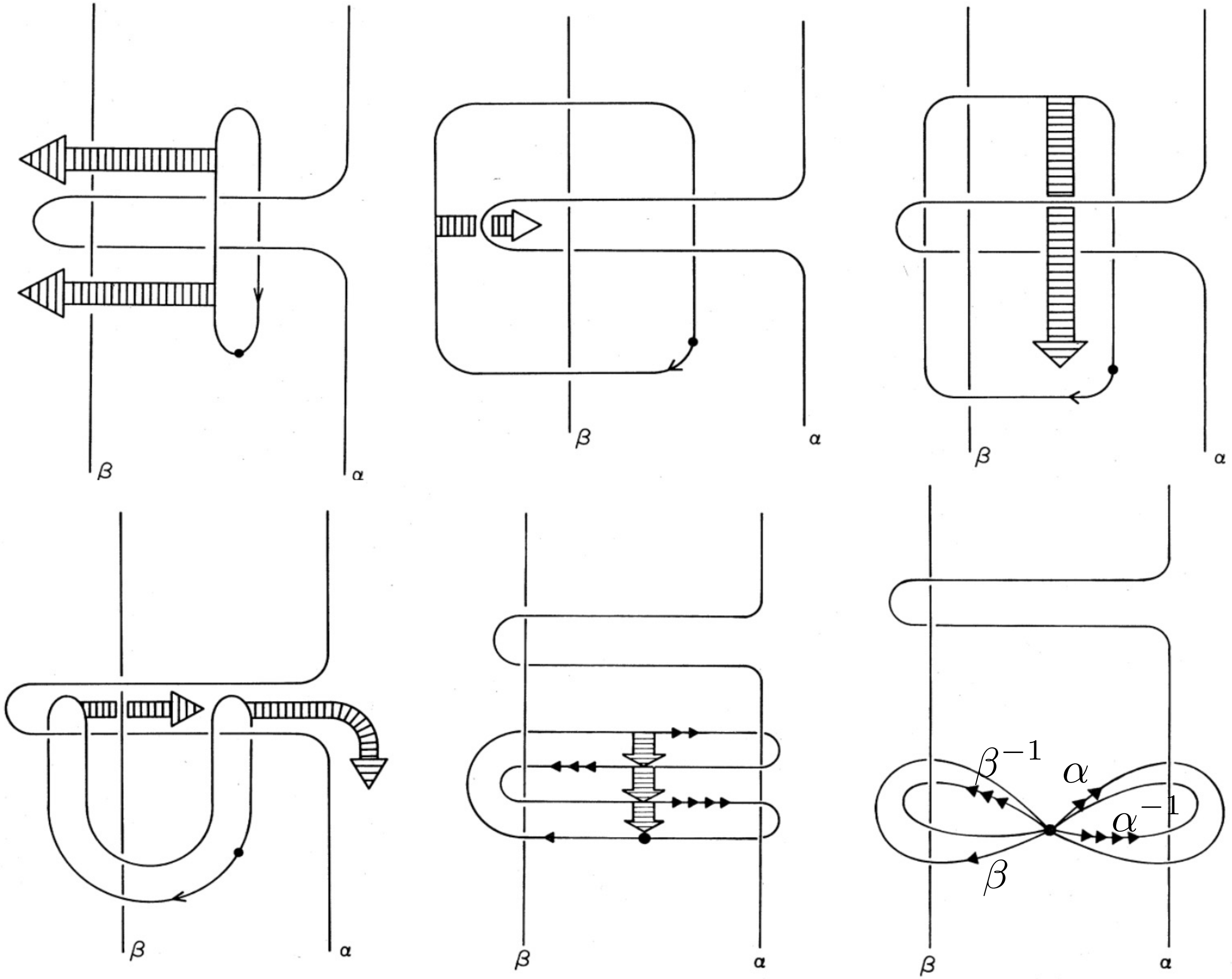


Pictorial Approach to Smooth Deformations

- If we take our contour around the crossing we can obtain the condition for scarring

$$\beta\alpha\beta^{-1}\alpha^{-1} \neq 1$$





VII. The Second Homotopy Group

Spheres from Squares: 1 pt Compactification

- The one point compactification of the square is the sphere
- This will enable a useful notation where spheres with the same base point are connected at the edges
- “Closing the purse”

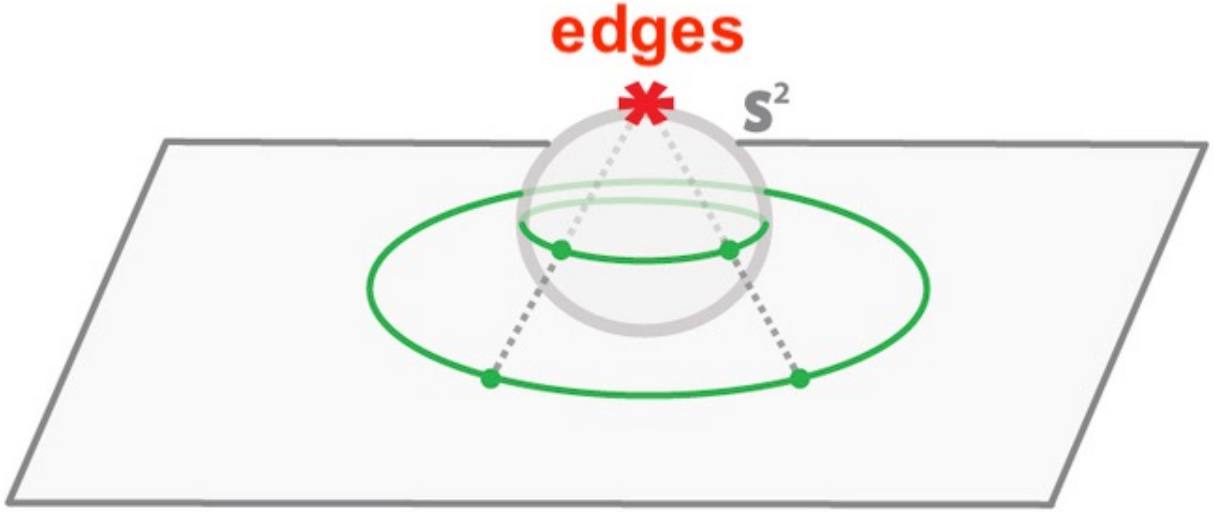
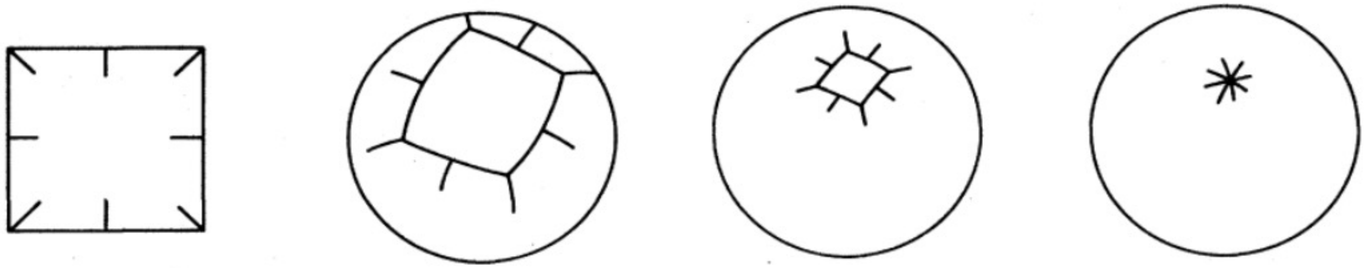
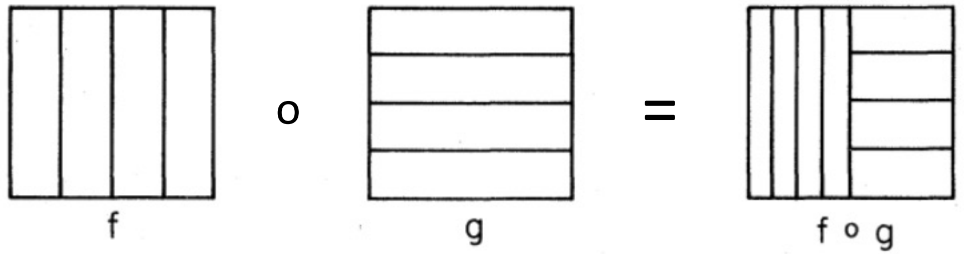


Image Credit: Adapted from Antoniou and Lambroupoulou

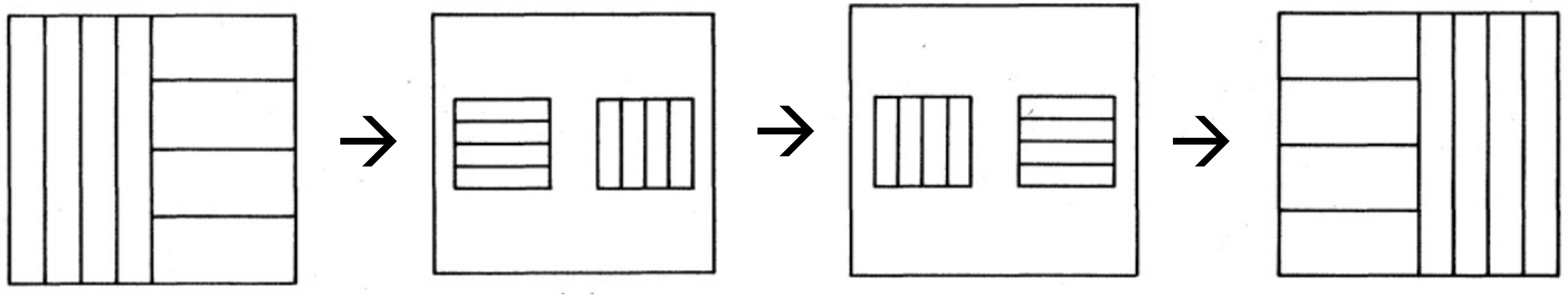


Theorem: π_2 is Abelian

- Composition of functions

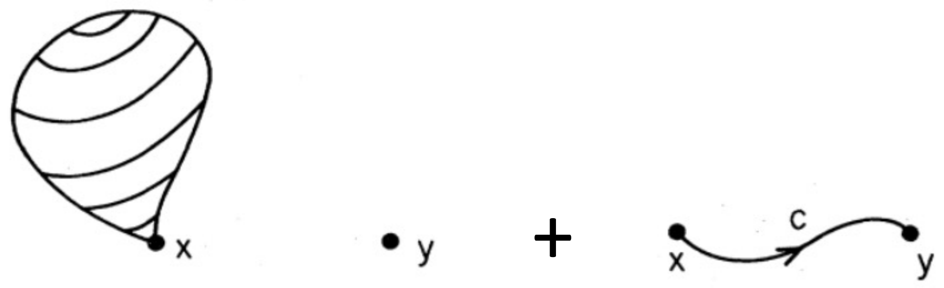


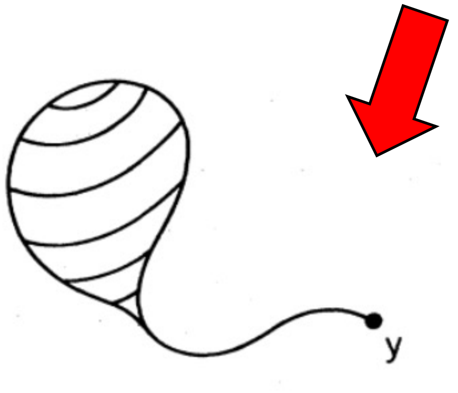
- Proof:



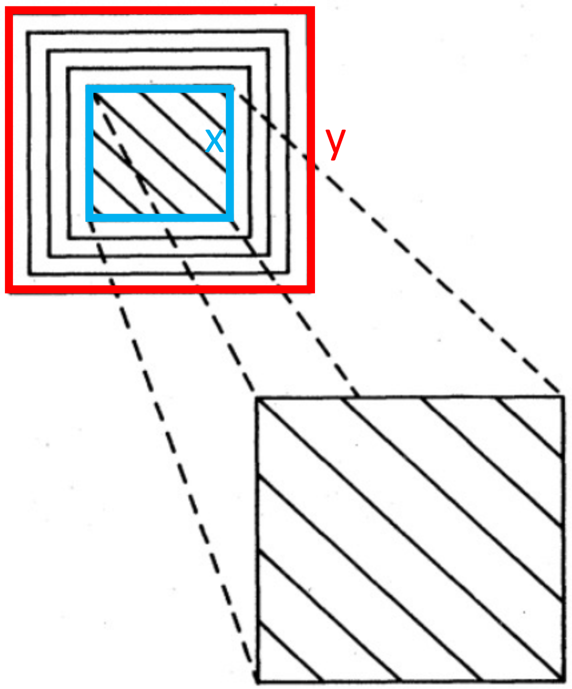
Theorem: π_2 is Base Point Independent

- Provided the spheres include the same volume
- Shifting to other base points requires that the enclosed region be **2-simple**, i.e. c induces a trivial automorphism on π_2





Q: Is closing the purse here the same as closing the purse there?



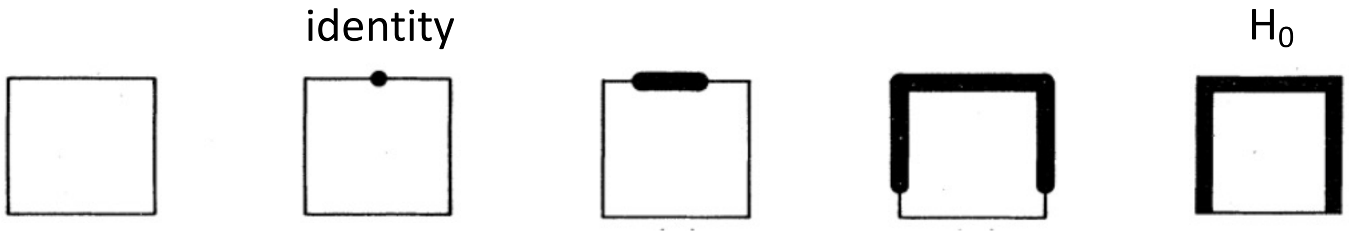
Fundamental Theorem of π_2

- Let G be simply connected (always possible—make it big enough)
- Let $\pi_2(G)=0$ (ok by Cartan's theorem for compact Lie groups)
- Let H be the isotropy subgroup (H = {g in G such that g Ψ = Ψ})
- The coset space is G/H
- Let H_0 be the connected component connected to the identity
- Then $\boldsymbol{\pi_2(G/H) = \pi_1(H_0)}$

- Proof: using the assumptions above establish an isomorphism between the two—one to one, and same algebraic structure

Pictorial Interpretation of the Fundamental Thm.

- Structure of H_0 on the edge of π_2 of the region



- Windings about H_0 gives us the possible “mouths” of the “purses”
- Here the $\pi_2(G)$ is assumed to be trivial so the only non-trivial behavior comes from H_0 and its winding about the “purse”

Exact Sequences

- This is an example of an exact sequence
- Since $\pi_2(G) = 0$ we can map to the kernel of H (H_0)
- This simplifies the computation of higher homotopy groups
- Exact sequences lead to the Bott periodicity theorem

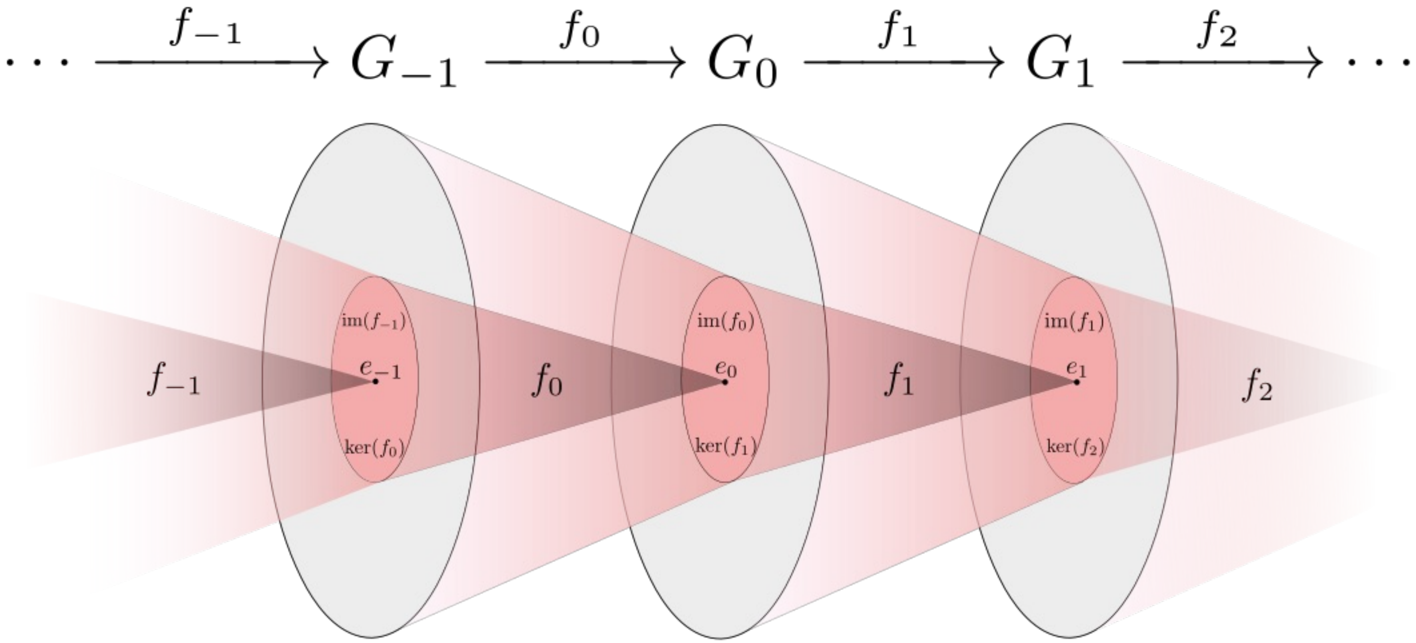


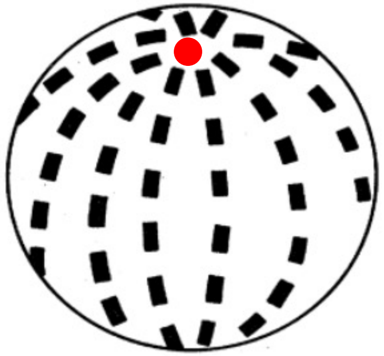
Image credit: Wikipedia

Corollary

- If H is discrete, then H_0 is a simple point
- Hence $\pi_1(H_0) = 0$
- So that $\pi_2(G/H) = \pi_1(H_0) = 0$

NO BIAXIAL HEDGEHOGS!

- Consequence
  - H is discrete for biaxial nematics and superfluid ^3He
  - These systems do not exhibit point defects in 3D



Ex. Spins and Nematics

- Spins
  - G = SU(2)
  - H = {rotations about one axis}
  - $H_0 = U(1)$
  - $\pi_2(G/H) = \pi_1(H_0) = \pi_1(U(1))$ = {winding numbers} = Z
- Nematics
  - The same but an additional symmetry
    - +n and -n winding numbers correspond to the same field configuration
  - $\pi_2(G/H) = Z^+$

Comparison of π_1 and π_2

System	π_1 (First Homotopy Group)	π_2 (Second Homotopy Group)
Planar spins	Z	Not sensible
Spins in 3D	0	Z
Nematics	Z/2Z	Z^+
Biaxial nematics	Quaternions	0
Superfluid ^3He	Z/2Z	0

Bonus. Applications to Q. Condensed Matter

Ex. Chern Numbers

- Consider a N x N Hamiltonian h (Hermitian matrix)
- We can decompose as $h = U D U^{-1}$
- Computation
  - G = U(N)
  - H = DU(N)
  - H0 = SDU(N)
  - $\pi_2(G/H) = \pi_1(H_0) = \pi_1(SDU(N)) = Z^N/Z = Z^{N-1}$
- Interpretation: N bands with integer "Chern" numbers c_n and $\Sigma c_n = 0$
- (for more details and related discussion see Moore, page 9)

Bott Periodicity Theorem

- From structure of exact sequences
- Unitary Groups
  - For N ≥ (n+1)/2

$$\pi_n(U(N)) \cong \pi_n(SU(N)) \cong \begin{cases} 1 & n \text{ even} \\ Z & n \text{ odd} \end{cases}$$

- Orthogonal Groups
  - For N ≥ n+2

$$\pi_n(O(N)) \cong \pi_n(SO(N)) \cong \begin{cases} 1 & n \equiv 2,4,5,6 \mod 8 \\ Z/2Z & n \equiv 0,1 \mod 8 \\ Z & n \equiv 3,7 \mod 8 \end{cases}$$

Periodic Table of Topological Insulators (& SC)

- This Bott periodicity appears shows up in the classification of (quadratic) mean field topological insulators/superconductors

Class	T	C	S	δ = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Image credit: Pulcu (2021)

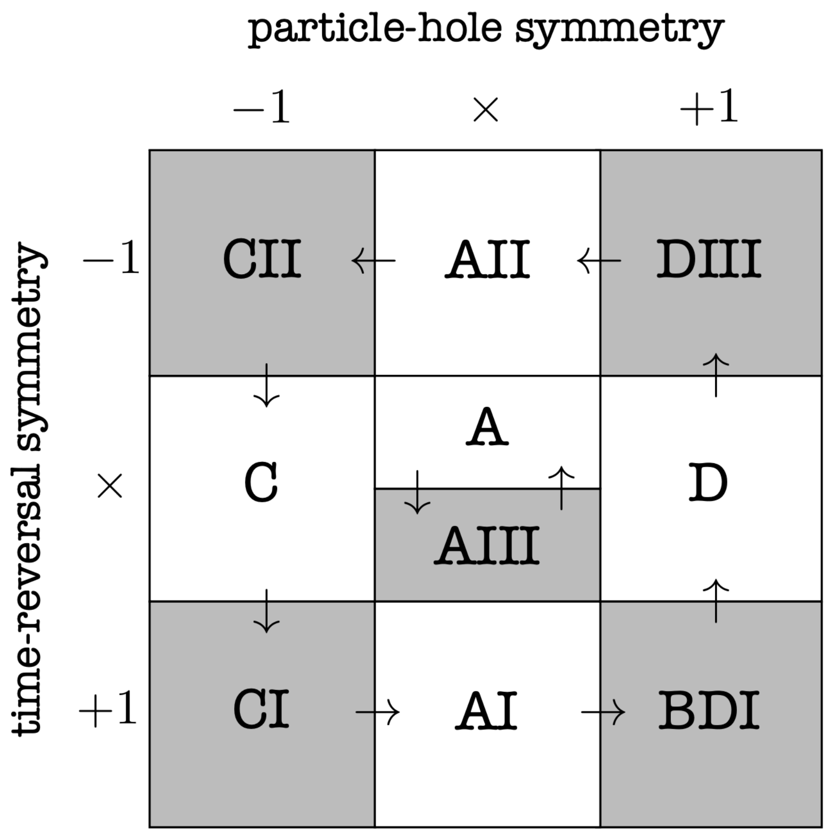


Image credit: Topology in Condensed Matter

Some Homotopy Groups

Table 4.1. Useful homotopy groups.

		π_1	π_2	π_3	π_4	π_5	π_6
SO(3)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SO(4)		$\mathbb{Z}_2$	0	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_{12}+\mathbb{Z}_{12}$
SO(5)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
SO(6)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SO(n)	$n > 6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
U(1)		$\mathbb{Z}$	0	0	0	0	0
SU(2)		0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SU(3)		0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$
SU(n)	$n > 3$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
S^2		0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^3		0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^4		0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
G_2		0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$
F_4		0	0	$\mathbb{Z}$	0	0	0
E_6		0	0	$\mathbb{Z}$	0	0	0
E_7		0	0	$\mathbb{Z}$	0	0	0
E_8		0	0	$\mathbb{Z}$	0	0	0

From Nakahara (2003)

References

- N.D. Mermin, *Topological theory of defects*, RMP 51, 591 (1979)
- J.E. Moore, *Notes for MIT minicourse on topological phases* (2011)
- Nakahara, *Geometry, Topology, and Physics*, 2nd Edition (2003)
- Topology in Condensed Matter (2015)