Continuum Field Theories for Fractons

Spenser Talkington ▪ 29 April 2022

Outline

(1) Intuition for fractons

(2) Continuum field theories
- ↳ Fractons usually don't fit into this picture

(3) Background: Magnetic monopoles

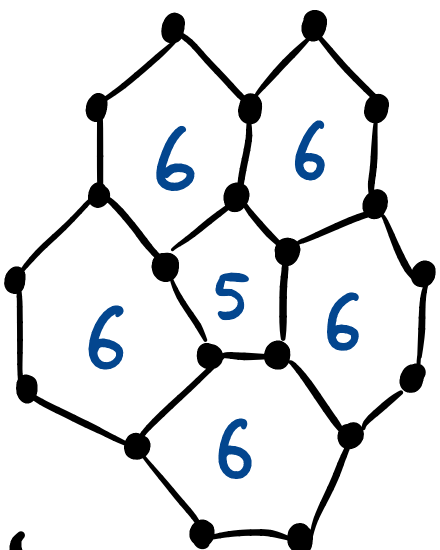
(4) $U(1)$ Tensor gauge theory
- ↳ The model and its analysis
- ↳ Fractons as defects in the theory

(5) Conclusion
- ↳ Other continuum models for fractons
- ↳ Outlook

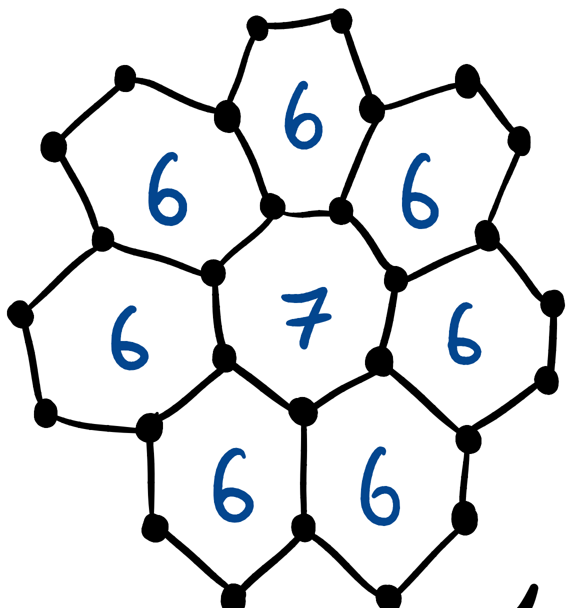
Ex. Lattice Defects

- Point defects in lattices

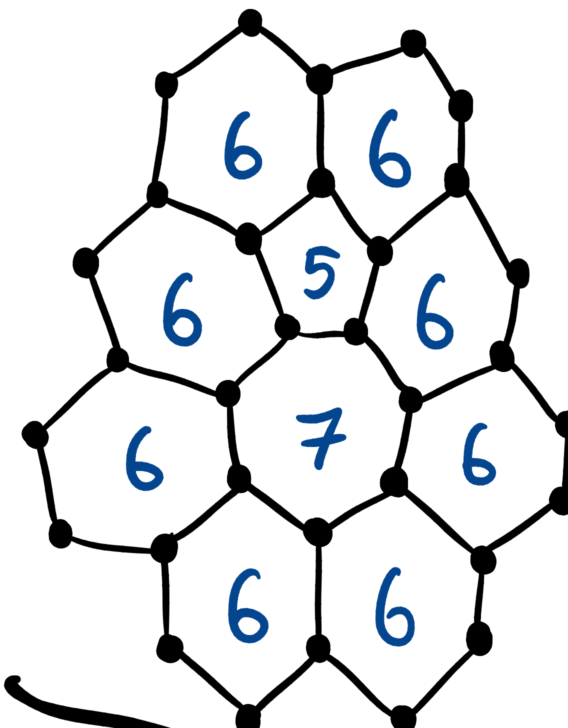
5 defect



7 defect



5-7 defect

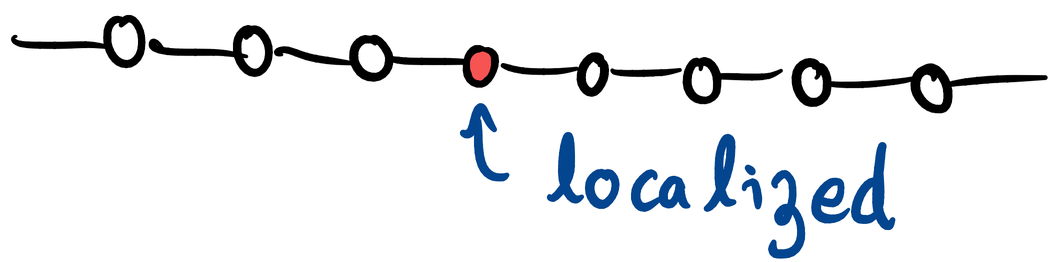


localized

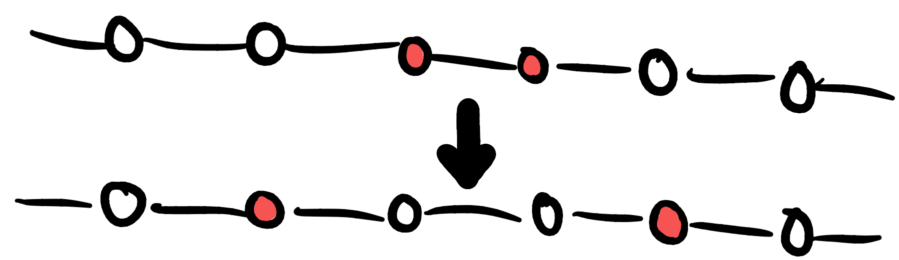
mobile

What Are Fractons?

- "Subdimensional particles"
  - ↳ Motion is restricted to a subspace
  - ↳ This restriction often originates with a conservation law
- Example: 1D lattice with conserved charge & dipole
  - ↳ 1 Charge



  - ↳ Multiple charges



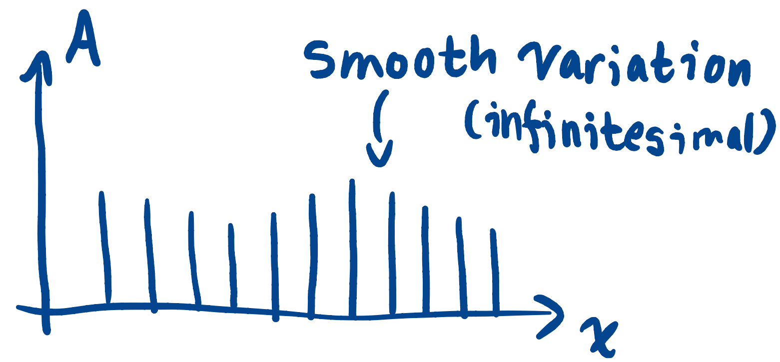
Multipole Conservation

Object	Maxwell	Tensor Maxwell
Vector Potential	A_i	A_{ij}
Canonical [,]	$[A_i(x), E_j(y)] = i\delta_{ij}\,\delta(x-y)$	$[A_{ij}(x), E_{kl}(y)] = i(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\delta(x-y)/2$
Gauss' Law	$\partial_i E_i = \rho$	$\partial_i \partial_j E_{ij} = \rho$
Charge	$Q = \int d^3x\, \partial_i E_i = \oint dn_i\, E_i$	$Q = \int d^3x\, \partial_i \partial_j E_{ij} = \oint dn_j\, \partial_i E_{ij}$
Dipole	Not conserved	$P_i = \int d^3x\, x_i \rho = \oint dn_k (x_i \partial_j E_{jk} - E_{ik})$ — P_i Conserved! — Stokes Thm.

Why Continuum FT?

- Enables us to transfer what we know for lattices to continuum models we may want to modify
- Low energy effective field theories are universal
  - ↳ Finite number of relevant operators
  - ↳ Taylor expansion

$$A(x+a) = \sum_{n=0}^{\infty} \frac{A^{(n)}(x)}{n!} a^n$$



  - ↳ Irrelevant operators are typically suppressed by powers of the lattice constant a
  - ↳ No UV-IR mixing

Breakdown of Continuum F.T.

- Let $A(x)$ be discontinuous
  - ↳ typical scenario for lattice field theories
  - ↳ fractons are localized which motivates making $A(x)$ discontinuous
- $A^{(n)}(x)$ can now be arbitrarily large ← $\frac{d}{dx}\Theta(x)=\delta(x)$
  - ↳ Higher order terms are not irrelevant for continuum F.T.
  - ↳ UV and IR mix
- Key question: Can we construct a continuum field theory that exhibits fractons?

Mathematical Detour

① non-trivial homotopy

Open covers of base space

transition functions

fiber bundle construction

→ Quantization of magnetic monopoles

② Unusual global symmetry

Open covers of base space

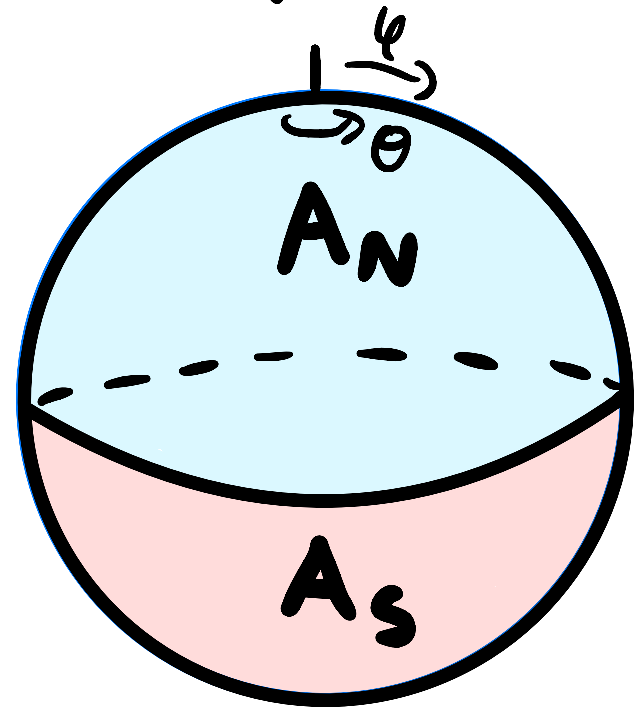
transition functions

fiber bundle construction

Gauge invariance

→ Fractons in the Continuum

Ex. Dirac Monopoles



- Dirac (Magnetic) monopoles emerge from the obstruction to a single smooth gauge
- Can have two gauges with

$$A_N = A_S + \nabla\alpha$$

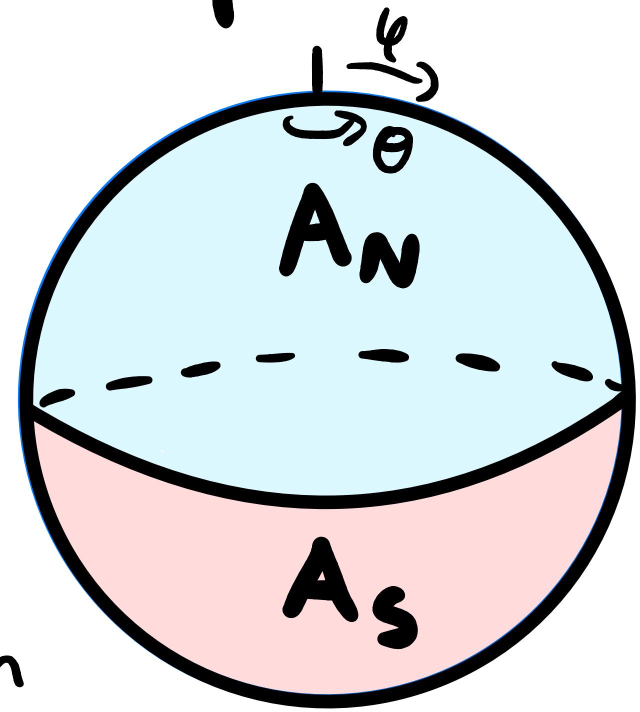
where $\alpha(\theta,\varphi)$ has an ambiguity from the periodicity of $\theta \in [0, 2\pi)$

- Example:

$$A_N = \frac{e}{4\pi R}\,\frac{1-\cos\varphi}{\sin\varphi} \quad \text{and} \quad A_S = -\frac{e}{4\pi R}\,\frac{1+\cos\varphi}{\sin\varphi}$$

with $\alpha = \frac{e}{2\pi}\theta$ or $\alpha = \frac{e}{2\pi}\theta\,\Theta(\pi/2 - \varphi)$

Ex. Dirac Monopoles



- This "obstruction" to a smooth choice of gauge field is the origin of the Dirac monopole

- The intersection of "open covers" is S^1 and $\pi_1(S^1)=\mathbb{Z}$ So we can construct a "non-trivial $U(1)$ fiber bundle"

- Ex. Partition S^2 into U_N, U_S s.t $U_N \cup U_S = S^2$. Now let $T_{NS}: U_N \cap U_S \to U(1)$ be the "transition function" e.g $T(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$ where n sets the charge

U(1) Tensor Gauge Theory

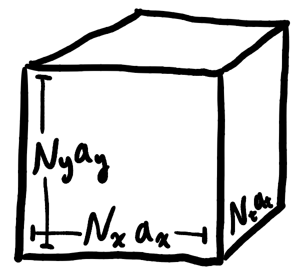
- Gauge field (A_t, A_{pos}) in 2+1 D

$A_{pos} := A_{xy} = A_{yx}$
$(A_{xx} = A_{yy} = 0)$

↳ $A_t \mapsto A_t + \partial_t \alpha$ and $A_{pos} \mapsto A_{pos} + \underbrace{\partial_x \partial_y}_{\text{two derivatives!}} \alpha$

- Gauge invariant electric field

↳ $E = \partial_t A_{pos} - \partial_x \partial_y A_t$



Space is $S^1 \times S^1 \times S^1$

- Lagrangian

↳ $\boxed{\mathcal{L} = \frac{1}{g^2} E^2 + \frac{\theta}{2\pi} E}$

θ is a "Witten effect" term that sets the "magnetic" charge of any fracton dyons

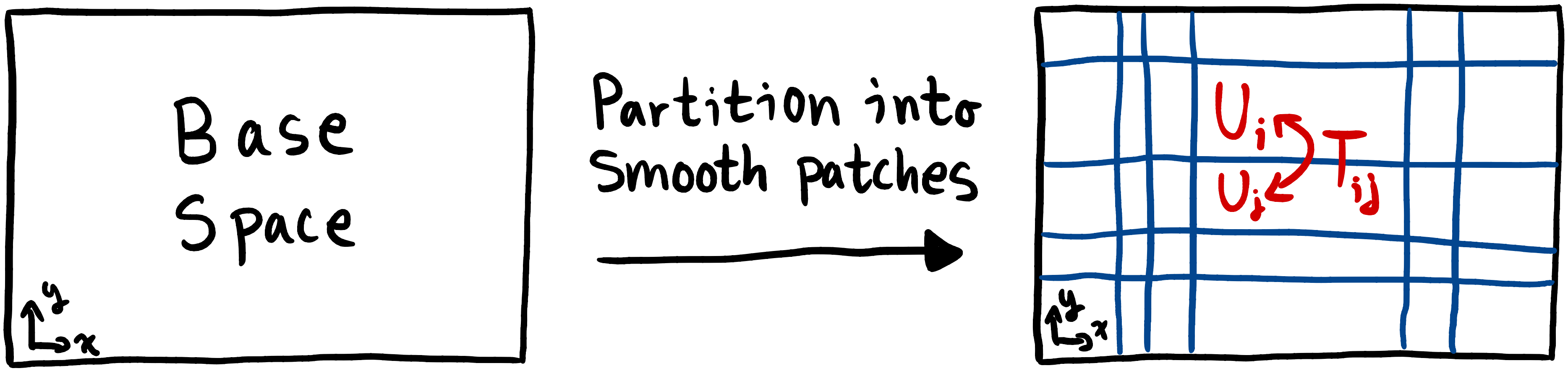
- Equations of motion

$\partial_t E = 0$ and $\partial_x \partial_y E = 0$

- Conserved current $J = 2E/g^2 + \theta/2\pi$

$\partial_t J = 0$ and $\partial_x \partial_y J = 0$

Piecewise Discontinuities



- Let the base space be $S^1 \times S^1 \times S^1$ and attach $U(1)$ fibers A_t and A_{pos} to each point (t, x, y)
- Let A be smooth and single valued on each patch and only have piecewise discontinuities given by non-trivial transition functions T_{ij} so that

$$A_j = A_i + \partial_x \partial_y T_{ij}$$

there is no A that is good everywhere.

Transition Functions

- Let $\{U_i\}$ be an "open cover" of the base space, then the transition functions $T_{ij}: U_i \cap U_j \rightarrow U(1)$ can be used to construct a fiber bundle.
- Generically, the "connection" in patch U_i is related to U_j by

$$A_j = T_{ij}^{-1} A_i T_{ij} + T_{ij}^{-1} \, d T_{ij}$$

- Here, we consider

$$A_j = A_i + \partial_x \partial_y T_{ij}$$

and let A_t and A_{pos} be single valued on each patch with transition functions of the form

$$T(x,y) = 2\pi \left(\frac{x}{N_x} \Theta(y-y_0) + \frac{y}{N_y} \Theta(x-x_0) - \frac{xy}{N_x N_y} \right)$$

Fluxes

- We consider the transition function (gauge transformation)

$$T(x,y) = 2\pi\left(\frac{x}{N_x}\Theta(y-y_0) + \frac{y}{N_y}\Theta(x-x_0) - \frac{xy}{N_xN_y}\right)$$

- So we have with $A_{pos}=0$ that $\tilde{A}_{pos} = A_{pos} + \partial_x\partial_y T$

$$\tilde{A}_{pos}(x,y) = 2\pi\frac{t}{a_t}\left(\frac{1}{N_x}\delta(y-y_0) + \frac{1}{N_y}\delta(x-x_0) - \frac{1}{N_xN_y}\right)$$

- From which we see

Recall: $E = \partial_t A_{pos} - \partial_x\partial_y A_t$

$$q_x(x) = \oint dt \oint dy\; E = 2\pi\,\delta(x-x_0)$$

$$q_y(y) = \oint dt \oint dx\; E = 2\pi\,\delta(y-y_0)$$

which give rise to the quantized fluxes (in $2\pi\mathbb{Z}$)

$$q_x(x_1,x_2) = \int_{x_1}^{x_2} dx\; q_x(x) \quad \text{and} \quad q_y(y_1,y_2) = \int_{y_1}^{y_2} dy\; q_y(y)$$

't Hooft Loop Operator

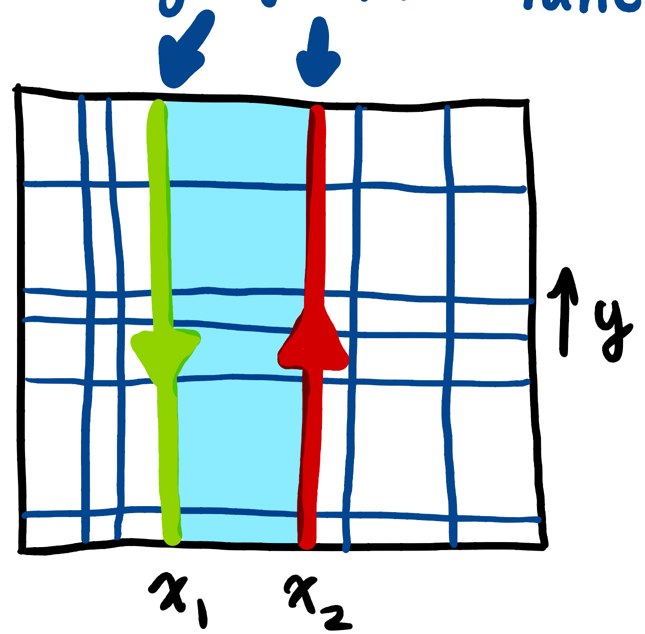
- Recall the conserved current is $J = 2E/g^2 + \theta/2\pi$ where $\partial_t J = 0$ and $\partial_x \partial_y J = 0$
- The gauge invariant objects are the 't Hooft - Wilson <u>strip</u> operators

The usual <u>lines</u> are charged and hence gauge variant

$$W_x(x_1, x_2) = e^{i\int_{x_1}^{x_2} dx \oint dy\, A_{pos}}$$
$$W_y(y_1, y_2) = e^{i\int_{y_1}^{y_2} dy \oint dx\, A_{pos}}$$

- To be gauge invariant the loop operators must be invariant under $A_{pos} \mapsto \tilde{A}_{pos}$
  ↳ this is only possible if (x_1, x_2) contains no lines of discontinuity

Cannot cross a discontinuity or W would not be gauge invariant

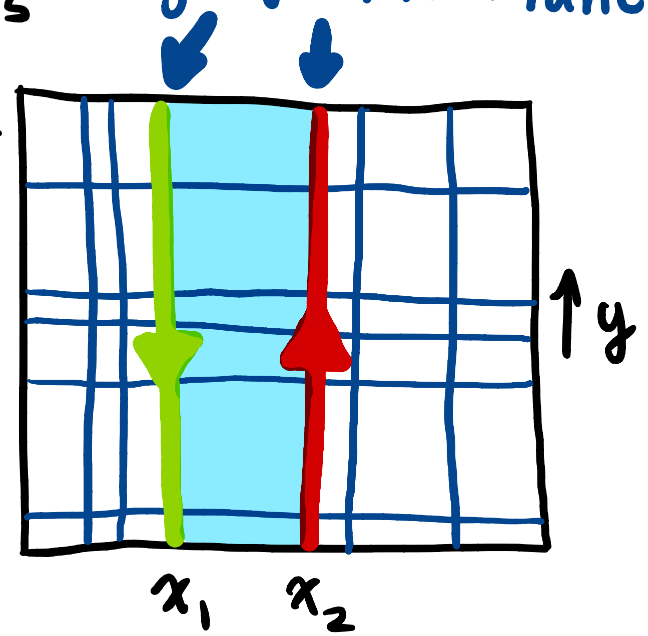


Fractons as Defects

- Monopole motion is restricted to 0D. Dipole motion is in 1D.
- The 't Hooft-Wilson strip operators are <u>the</u> gauge-invariant operators that generate the various ground states of the theory

$$|\Omega'\rangle = W|\Omega\rangle$$

Cannot cross a discontinuity or W would not be gauge invariant



- Physically, W is generating a dipole-anti-dipole pair, winding one around the torus and annihilating
- The ground-state degeneracy goes is $N_x + N_y - 1$ which is not a constant or a volume law!

Summary: U(1) Model

- Gauge invariance can restrict the mobility of excitations
- The U(1) tensor model realizes key features of fractons:
  - ↳ fractional mobility
  - ↳ exotic global (dipole) symmetry
  - ↳ unusual ($\sim V^{1/D}$) scaling of ground state degeneracy
- The restrictions originate from discontinuities in the gauge field described by "transition functions" that construct a non-trivial fiber bundle in analogy to the non-trivial bundle of the Dirac monopole

Other Models

- Many lattice and a few continuum models
- Common features
  - Localization
  - Unusual (tensor) global symmetries
  - Unusual ground-state degeneracy
- X-Cube model: the paradigmatic lattice model
- ϕ model in 1+1D

$$\mathcal{L} = \frac{\mu_t}{2}(\partial_t \phi)^2 - \frac{1}{2\mu}(\partial_x \partial_y \phi)^2 \Rightarrow \omega^2 = p_x^2 p_y^2$$

- $\mathbb{Z}_N$ model in 2+1D: simple extension of U(1) model

$$\mathcal{L} = \frac{N}{2\pi} \phi E$$

Conclusion

- Fractons broaden the phenomenology we should expect to describe using continuum field theories
- This phenomenology is rich, can extend many models and has many generalizations: e.g. different fibers/connections, time-localized fractons, and matter coupling
- Behavior characteristic of fractons can emerge in a simple U(1) tensor continuum field theory
- The presence of fracton-like defects in real crystal lattices is well established

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