Fractional Quantum Hall Effect via Chern Simons Theory

Spenser Talkington ▪ 14 December 2021

Outline

(1) Experiments

(2) Postulates and possible terms in $\mathcal{L}$

↳ Chern-Simons term $\mathcal{L}_{cs}$

(3) When is $\mathcal{L}_{cs}$ gauge invariant?

(4) Consequences of $\mathcal{L}_{cs}$

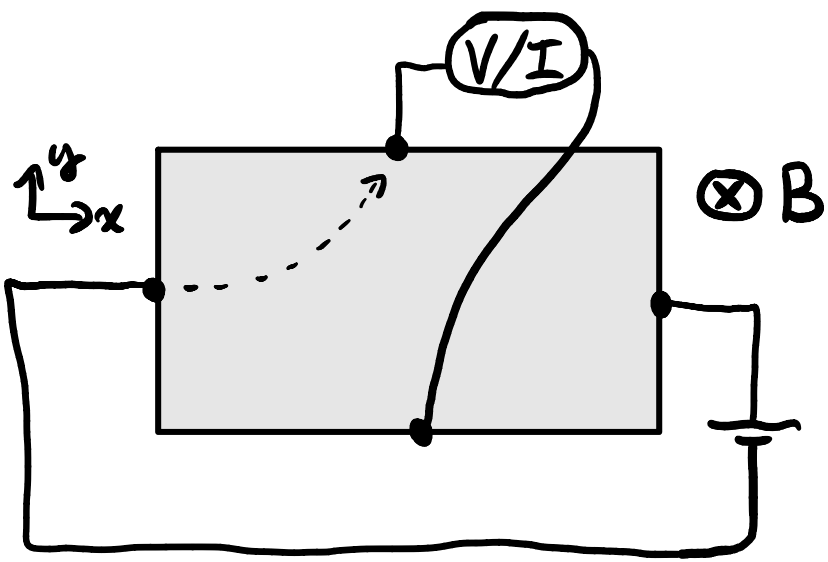
↳ gapped spectrum

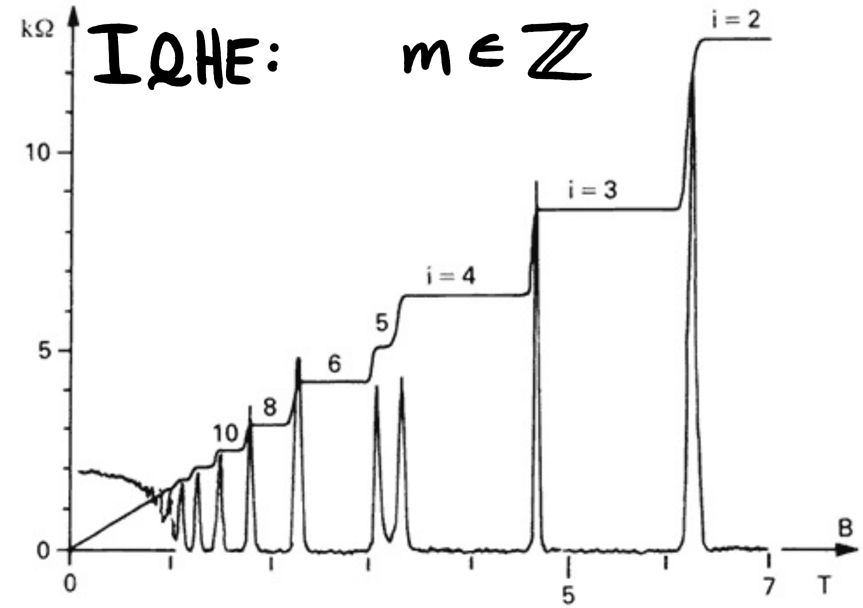
↳ quantized resistance

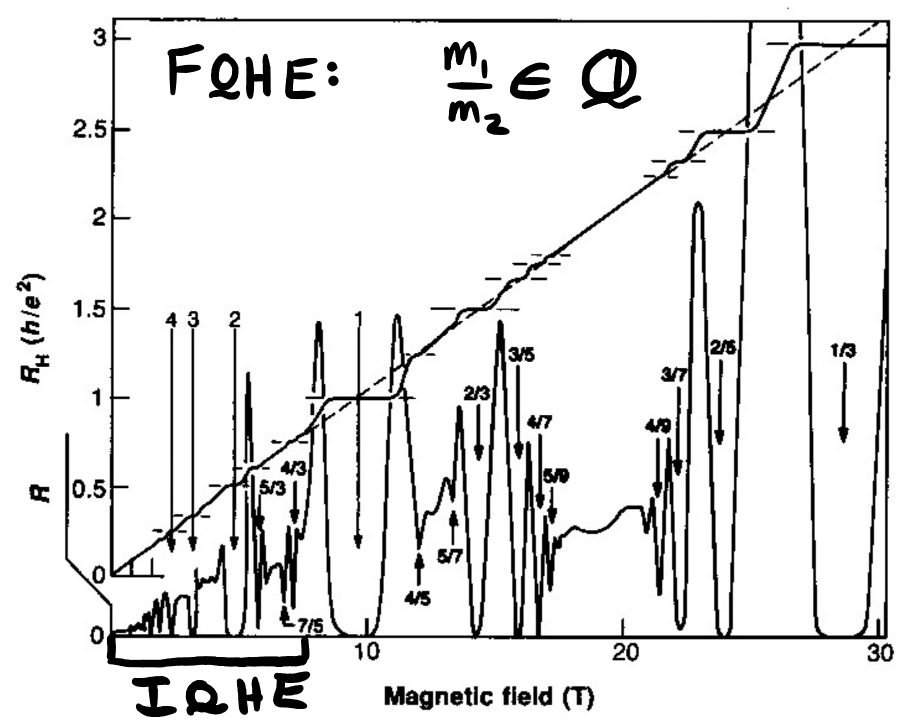
Experiments

$$R_{xx} \to 0$$

$$R_{xy} \to m\frac{h}{e^2} \text{ and } \frac{m_1}{m_2}\frac{h}{e^2}$$

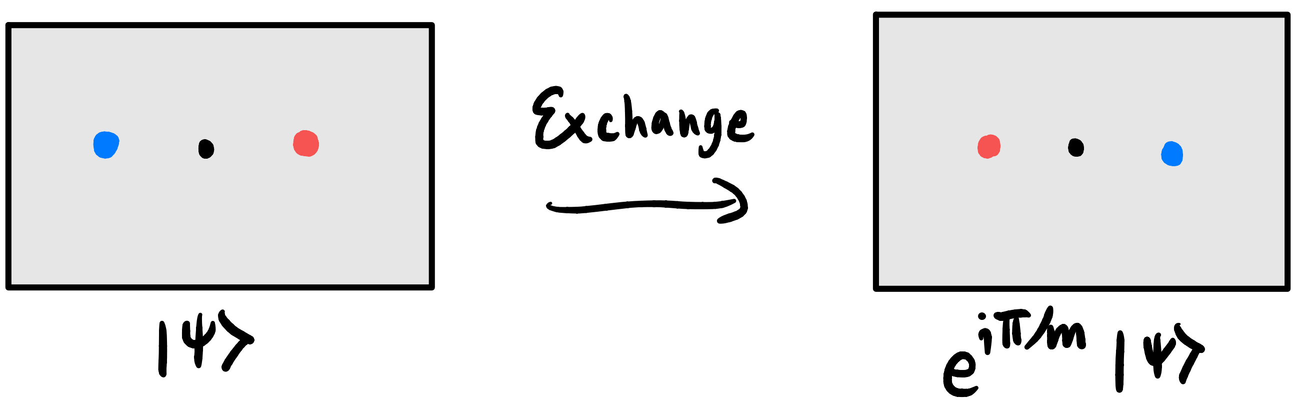






Why You Should Care

- Fractionalized Charge
  - ↳ quasiparticles with conductance $\frac{1}{m}\frac{h}{e^2}$ have charge $\frac{e}{m}$
- Fractional Statistics
  - ↳ quasiparticles with conductance $\frac{1}{m}\frac{h}{e^2}$ have fractional exchange statistics: $\theta = \frac{\pi}{m}$



Current Response

- Real gauge field couples to the current J

$$S_A = \int dt\, d^2x\ J_\mu A^\mu$$

- The corresponding conserved current is $\partial_\mu J^\mu = 0$

- Extremize action, then

$$\frac{\delta S_A}{\delta A_\mu} = J^\mu$$

- Hall conductivity $\Longrightarrow$ Hall resistance

$$J^x = \sigma^{xy} E_y \qquad\qquad R_{xy} = \sigma_{xy}^{-1}$$

Field Theory: Coarse Graining

- Impose symmetries and general considerations
  - ↳ without attention to microscopic details

- To explain the FQHE, we need only assume
  - ↳ At *low* energy there are no D.O.F.s that change the perturbative response

    $\Rightarrow$ There is a gap!

  - ↳ For convenience, also assume rotational symmetry

- Study the low energy and long range behavior

Perturbative Real Gauge Field

- A_μ is ostensibly large O(10 Tesla)
- But we are not interested in electronic motion
  - ↳ quasiparticle motion
  - ↳ introduce an <u>emergent</u> gauge field a_μ
- Add all symmetry-allowed terms
- The coupling between A_μ and a_μ is perturbative

Actions

- Rotational symmetry
  $\Longrightarrow$ only an even number of gauge fields
- Low energy/Long range
  $\Longrightarrow$ only lowest order terms/derivatives
- The allowed terms up to second order

$$\mathcal{L} = \underbrace{\mathcal{L}_{linear}}_{\substack{\text{one field/}\\ \text{derivative}}} + \underbrace{\mathcal{L}_{maxwell}}_{\substack{\text{two fields/}\\ \text{two derivatives}}} + \underbrace{\mathcal{L}_{chern\text{-}simons}}_{\substack{\text{one field \underline{and}}\\ \text{one derivative}}}$$

Remaining Terms

- Chern-Simons action

$$S_{cs}[A,a] = \frac{e^2}{h}\int dt\, d^2x\, \mathcal{L}_{cs}$$

$$\mathcal{L}_{cs} = -\frac{m_A}{2}\varepsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}A^{\rho} + \varepsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}a^{\rho} - \frac{m_a}{2}\varepsilon_{\mu\nu\rho}a^{\mu}\partial^{\nu}a^{\rho}$$

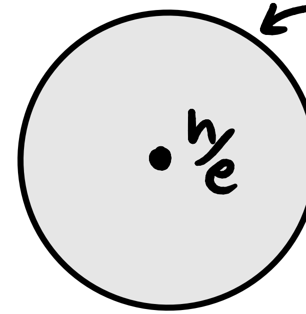
- $\mathcal{L}_{cs}$ breaks parity and TRS
- But is $\mathcal{L}_{cs}$ gauge invariant?
  - ↳ At first glance, yes:

$$A_\mu \mapsto A_\mu + \partial_\mu \omega$$

$$\mathcal{L}^{A}_{cs} \mapsto \mathcal{L}^{A}_{cs} - \frac{m_A}{2}\partial^{\mu}(\omega\, \varepsilon_{\mu\nu\rho}\partial^{\nu}A^{\rho})$$

Quantization of m (1/3)

- Thought experiment: what is S_{cs} on a sphere S^2 surrounding a Dirac monopole?



- We have $\int d^2x\, F_{xy} = \frac{h}{e}$
- The CS action is

$$S_{cs} = -\frac{e^2}{h}\frac{m_A}{2}\int dt\, d^2x\, A_0 F_{xy} + A_x F_{y0} + A_y F_{0x}$$

$$= -\frac{e^2}{h}\frac{m_A}{2}\int dt\, d^2x\, 2A_0 F_{xy}$$

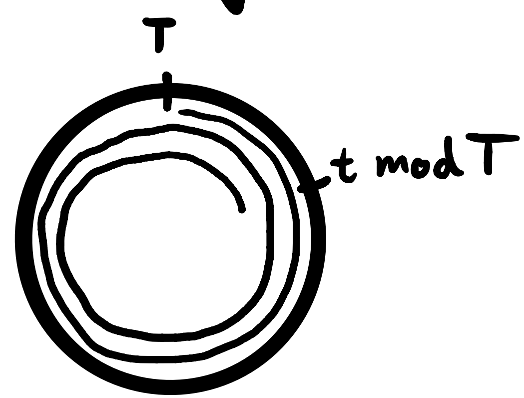
$$= -\frac{e^2}{h}\frac{m_A}{2}\, 2A_0\, T\frac{h}{e}$$

$$= -e\, m_A A_0 T$$

Quantization of m (2/3)

- How does S_{cs} change under gauge transformation?
- Generally $A_\mu \mapsto A_\mu + \partial_\mu \omega \Rightarrow \phi \mapsto e^{ie\omega}\phi$
  - ↳ ω need not be single valued
  - ↳ only $e^{ie\omega}$ needs to be
  - ↳ so called "Large" gauge transformation
- With time-periodicity T, this means

$$\omega = \frac{2\pi}{e}\frac{t}{T}$$



- A_0 transforms as

$$A_0 \mapsto A_0 + \frac{2\pi}{e}\frac{1}{T}$$

Quantization of m (3/3)

- We have
  - $\hookrightarrow$ $S_{cs} = -e\, m_A A_0 T$
  - $\hookrightarrow$ $A_0 \mapsto A_0 + \frac{2\pi}{e}\frac{1}{T}$
- So under gauge transformation

$$S_{cs} \mapsto S_{cs} - e\, m_A \frac{2\pi}{e}\frac{1}{T} T$$
$$= S_{cs} - 2\pi m_A$$

- Observables come from the partition function

$$Z = e^{iS_{cs}} \mapsto e^{iS_{cs}} e^{-i2\pi m_A}$$

- Observables <u>only</u> invariant if $m_A \in \mathbb{Z}$

Integer QHE

- We have the Chern-Simons Action

$$S_{cs}[A] = -\frac{e^2}{h}\int dt\, d^2x\, \frac{m_A}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho$$

- We find the current by the functional derivative

$$J_i = \frac{\delta S_{cs}}{\delta A_i} = -\frac{e^2}{h}\frac{m_A}{2}2\,\varepsilon_{0ij}A^j$$

- Which is

$$\sigma_{xy} = m_A\frac{e^2}{h}$$

Fractional QHE (1/2)

- Consider the emergent gauge field a_μ and its coupling to A_μ

$$S_{cs}[A,a] = \frac{e^2}{h}\int dt\, d^2x \; \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu a^\rho - \frac{m_a}{2} \varepsilon_{\mu\nu\rho} a^\mu \partial^\nu a^\rho$$

- Equations of motion

$$\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} - m_a \underbrace{(\partial_\mu a_\nu - \partial_\nu a_\mu)}_{f_{\mu\nu}} = 0$$

$$\Longrightarrow f_{\mu\nu} = \frac{1}{m_a} F_{\mu\nu} \Longrightarrow a_\mu = \frac{A_\mu}{m_a}$$

- Effective action is

$$S_{eff}[A] = \frac{e^2}{h}\int dt\, d^2x \; \frac{1}{2m_a} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$$

Fractional QHE (2/2)

- Effective action

$$S_{eff}[A] = \frac{e^2}{h}\int dt\, d^2x\, \frac{1}{2m_a}\,\varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$$

- We find the current by the functional derivative

$$J_i = \frac{\delta S_{cs}}{\delta A_i} = \frac{e^2}{h}\frac{1}{2m_a} 2\,\varepsilon_{0ij} A^j$$

- Which is

$$\sigma_{xy} = \frac{1}{m_a}\frac{e^2}{h}$$

But, is it Gapped?

- Consider quasiparticle motion alone

$$S^a = S^a_{maxwell} + S^a_{cs}$$

$$= \int dt\, d^2x \; \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} - \frac{e^2}{h}\frac{m_a}{2} \varepsilon_{\mu\nu\rho}\, a^\mu \partial^\nu a^\rho$$

- Equation of motion

$$\frac{1}{g^2}\partial_\mu f^{\mu\nu} + \frac{e^2}{h}\frac{m_a}{2}\varepsilon^{\nu\rho\sigma} f_{\rho\sigma} = 0$$

- This describes a field theory with a massive photon
  - ↳ There is a gap

$$E_{gap} = g^2 \frac{e^2}{h}\frac{m_a}{4}$$

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- Fractionalized Charge
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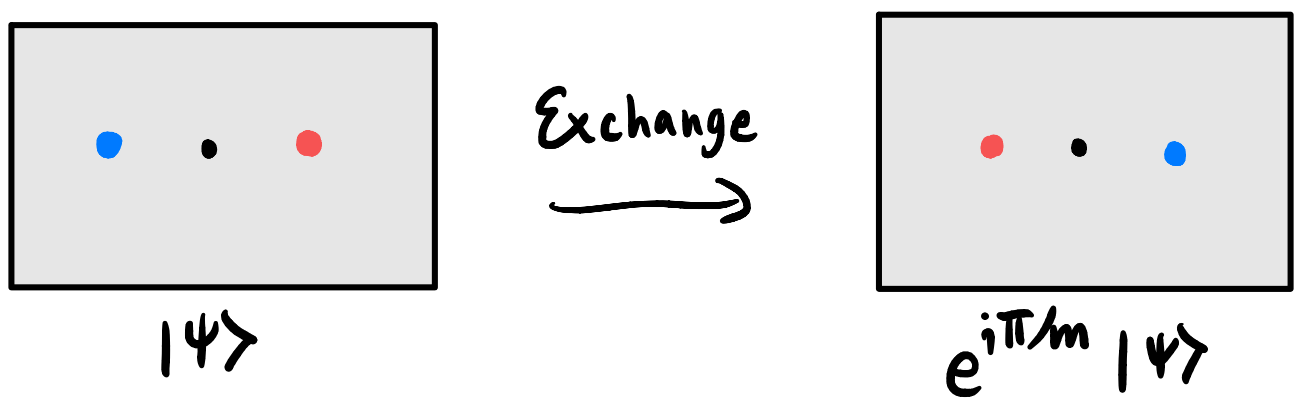

Questions?

Other FQHE States

- Introduce <u>more</u> emergent gauge fields
  - ↳ Simplest case a_1 & a_2 $\Rightarrow$ $\dfrac{1}{m_1 - \frac{1}{m_2}}$ State, etc.
- "k matrix" generalization

$$S_{cs} = \frac{e^2}{h} \int dt\, d^2x \sum_i \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu a_i^\rho - \sum_{i,j} \frac{K_{ij}}{2} \varepsilon_{\mu\nu\rho}\, a_i^\mu \partial^\nu a_j^\rho$$

$\Longrightarrow$ $\sigma_{xy} = \mathrm{Tr}(K^{-1}) \dfrac{e^2}{h}$

$\Longrightarrow$ Exchange statistics $\theta_{ij} = (K^{-1})_{ij}\, \pi$