Definitions

- A **parameterized curve** is a function $\vec{\alpha}: I \subseteq \mathbb{R} \mapsto \mathbb{R}^n$.
- A curve, $\vec{\alpha}(t)$ is **regular** iff $\vec{\alpha}'(t) \neq \vec{0}$ for all $t \in I$.
- A **smooth function**, or $\mathscr{C}^\infty$ function, is a function whose derivatives are continuous at all orders.
- Two curves $\vec{\alpha}$ and $\vec{\beta}$ are **parameterization equivalent** iff there exists a smooth invertible function s where $\vec{\alpha}(t) = \vec{\beta}(s(t))$.
- A curve $\vec{\alpha}(t)$ is an **arclength parameterization** iff $||\vec{\alpha}'(t)|| = 1\ \forall t$. This is also called a **unit-speed curve** and is indicated by $\vec{\alpha}(s)$.
- Two curves are **congruent** iff they differ by a rotation and a translation.

Elementary Relations

The dot product and cross product and their geometric interpretations are assumed to be familiar. Derivatives are:

$$\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}$$

$$\frac{d}{dt}(\vec{r}\times\vec{s}) = \frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}$$

The angle between vectors arises in the dot product (**orthogonal** if $\vec{u}\cdot\vec{v} = 0$):

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos(\theta)$$

Point-normal equation for a plane:

$$n_x(x-p_x)+n_y(y-p_y)+n_z(z-p_z) = 0$$

The derivative of a curve is as expected: if $\vec{\alpha}(t) = (\alpha_x(t), \alpha_y(t), \alpha_z(t))$, then $\vec{\alpha}'(t) = (\alpha_x'(t), \alpha_y'(t), \alpha_z'(t))$.

The magnitude of a curve at time t is

$$||\vec{\alpha}(t)|| = \sqrt{\alpha_x^2(t) + \alpha_y^2(t) + \alpha_z^2(t)}$$

The equation of the tangent line $\vec{\mathcal{T}}$ to a curve at $\vec{\alpha}(t_0)$ is $\vec{\mathcal{T}}(u) = \vec{\alpha}(t_0) + u\vec{\alpha}'(t_0)$ with $\vec{\mathcal{T}}$ is parameterized by $u \in J \subseteq \mathbb{R}$.

The arclength function is:

$$s(t, t_0) = \int_{t_0}^t du\ ||\vec{\alpha}'(u)||$$

Foundational Theorems

Thm. If $\vec{\alpha}$ and $\vec{\beta}$ are **parameterization equivalent**, then the image of $\vec{\alpha}$ and the image of $\vec{\beta}$ are equal.

Thm. If the arclength function $s(t,0)$ is smooth and invertible with smooth inverse $t := s^{-1}(s(t,0))$, then $\vec{\beta}(s) := \vec{\alpha}(t(s))$ is parameterization equivalent to $\vec{\alpha}(t)$ and $\vec{\beta}(s)$ unit speed.

Thm. If $\vec{\alpha}: [a,b] \to \mathbb{R}^n$ is a smooth, regular curve, then $\vec{\alpha}$ can be reparameterized with respect to arclength.

Thm. If $||\vec{v}|| = \text{const}$, then $\vec{v}(t) \perp \vec{v}'(t)$.

Frenet-Serret Apparatus

For any smooth regular curve in $\mathbb{R}^3$, we have the unit **tangent**, **normal**, and **binormal** which form an orthonormal basis for $\mathbb{R}^3$, the **Frenet frame**:

$$\vec{T}(t) = \frac{\vec{\alpha}'(t)}{||\vec{\alpha}'(t)||}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{(\vec{\alpha}'(t)\times\vec{\alpha}''(t))\times\vec{\alpha}'(t)}{||\vec{\alpha}'(t)\times\vec{\alpha}''(t)||\,||\vec{\alpha}'(t)||}$$

$$\vec{B}(t) = \vec{T}(t)\times\vec{N}(t) = \frac{\vec{\alpha}'(t)\times\vec{\alpha}''(t)}{||\vec{\alpha}'(t)\times\vec{\alpha}''(t)||}$$

and the **curvature** and **torsion**:

$$\kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{\alpha}'(t)||} = \frac{||\vec{\alpha}'(t)\times\vec{\alpha}''(t)||}{||\vec{\alpha}'(t)||^3}$$

$$\tau(t) = \frac{(\vec{\alpha}'(t)\times\vec{\alpha}''(t))\cdot\vec{\alpha}'''(t)}{||\vec{\alpha}'(t)\times\vec{\alpha}''(t)||^2}$$

The Frenet equations are:

$$\begin{pmatrix}\vec{T}'\\ \vec{N}'\\ \vec{B}'\end{pmatrix} = ||\vec{\alpha}'||\begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}$$

These expressions simplify when the curve is arclength parameterized:

$$\vec{T}(s) = \vec{\alpha}'(s)$$

$$\vec{N}(s) = \frac{\vec{\alpha}''(s)}{||\vec{\alpha}''(s)||}$$

$$\vec{B}(s) = \vec{T}(s)\times\vec{N}(s)$$

$$\kappa(s) = ||\vec{\alpha}''(s)||$$

$$\tau(s) = \vec{B}(s)\cdot\vec{N}'(s)$$

and the Frenet equations are the same, but with $||\vec{\alpha}'|| = 1$.

Osculating Planes/Circles

Three planes are defined as follows:

Name of plane	Spanned by
Osculating plane	$\vec{T}$ and $\vec{N}$
Rectifying plane	$\vec{T}$ and $\vec{B}$
Normal plane	$\vec{N}$ and $\vec{B}$

Def. The **osculating circle** is the circle with radius $1/\kappa(s_0)$ centered at $\vec{\beta}(s_0) = \vec{\alpha}(s_0) + \vec{N}(s_0)/\kappa(s_0)$. This is the best fit circle to $\vec{\alpha}(s)$ at $\vec{\alpha}(s_0)$.

Lines Through Points

Thm. If all the normal lines to $\vec{\alpha}(s)$ pass through a single point, then $\vec{\alpha}$ has constant curvature and zero torsion.

Thm. If all tangent lines to $\vec{\alpha}(s)$ pass through a single point, then $\vec{\alpha}$ is a line.

Thm. If all osculating planes of $\vec{\alpha}(s)$ pass through a single point, then the curve is planar.

Lines

Def. $\vec{\alpha}(s)$ is a **line** iff there exist constant vectors $\vec{p}$ and $\vec{T}$ such that $\vec{\alpha}(s) = \vec{p} + s\vec{T}$ for all s.

Thm. If $\vec{\alpha}''(t) = \vec{0}$ for all $t \in I$, then $\vec{\alpha}(t)$ is a line. Equivalently, $\vec{\alpha}(t)$ is a line iff $\kappa(t) \equiv 0$.

Planar Curves

Def. $\vec{\alpha}(s)$ is a **planar curve** iff there exist constant vectors $\vec{n}$ and $\vec{p}$ such that $(\vec{\alpha}(s) - \vec{p})\cdot\vec{n} = 0$ for all s.

Thm. The following are equivalent for a unit-speed curve $\vec{\alpha}(s)$ with $\kappa(s) > 0$:

1. $\vec{\alpha}(s)$ is a planar curve
2. $\vec{B}(s)$ is a constant vector
3. $\tau(s) \equiv 0$

Circles

Def. $\vec{\alpha}(s)$ is part of a **circle** iff there exists a constant vector $\vec{p}$ and a constant scalar r such that $||\vec{\alpha}(s) - \vec{p}|| = r$ and $\vec{\alpha}(s)$ is a planar curve.

Thm. For a unit-speed curve $\vec{\alpha}(s)$ with $\kappa(s) > 0$, $\vec{\alpha}(s)$ is a part of a circle iff $\kappa(s)$ is a constant and $\tau(s) \equiv 0$.

Generalized Helixes

Def. $\vec{\alpha}(s)$ is a **generalized helix** iff there exists a constant unit vector $\vec{A}$ and a constant scalar θ such that $\vec{T}(s)\cdot\vec{A} = \cos(\theta)$ for all s.

Thm. For a unit-speed curve $\vec{\alpha}(s)$ with $\kappa(s) > 0$, $\vec{\alpha}(s)$ is a generalized iff $\tau(s)/\kappa(s)$ is a constant.

Involutes and Evolutes

We say $\vec{\beta}$ is an **involute** of $\vec{\alpha}$ and $\vec{\alpha}$ is an **evolute** of $\vec{\beta}$ if for each $t \in I$ we have: $\vec{\beta}(t)$ lies on the tangent line to $\vec{\alpha}$ at $\vec{\alpha}(t)$ and $\vec{\alpha}'(t)$ is orthogonal to $\vec{\beta}'(t)$.

Thm. $\vec{\beta}(s)$ is involute of $\vec{\alpha}(s)$ iff $\vec{\beta}(s) = \vec{\alpha}(s) + (c-s)\vec{T}(s)$ for some constant c.

Fundamental Theorems

Fundamental theorem of paths: any smooth, regular, arclength parameterized curve with $\kappa(s) > 0$ is completely determined by curvature κ and torsion τ up to initial position and direction.

Fundamental theorem of space curves: let $\vec{\alpha}(s)$ and $\vec{\beta}(s)$ be two unit-speed curves with curvature $\kappa_\alpha(s) = \kappa_\beta(s)$ for all s. Then there exists a translation and a rotation that takes $\vec{\alpha}$ to $\vec{\beta}$.

Surfaces

Def. $\vec{x}: U \subseteq \mathbb{R}^2 \to M \subseteq \mathbb{R}^3$ is a **regular parameterization** (of a surface) if $\vec{x}$ is an injective $\mathscr{C}^3$ function and $\vec{x}_u \times \vec{x}_v \neq \vec{0}$ for all $(u,v) \in U$.
Def. A connected subset $M \subseteq \mathbb{R}^3$ is a **surface** iff each point $p \in M$ has a neighborhood surrounding it that is regularly parameterized.

Parameterized Surfaces

The **tangent plane** at any point on a regular surface is the plane spanned by $\vec{x}_u$ and $\vec{x}_v$, and the unit normal is:

$$\vec{n} = \frac{\vec{x}_u \times \vec{x}_v}{||\vec{x}_u \times \vec{x}_v||}$$

Surfaces of Revolution

The **surface of revolution** of $\vec{\alpha} = (0, f(u), g(u))$ about the z-axis is:
$\vec{x}(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$

Ruled Surfaces

Let $\vec{\alpha}(u)$ and $\vec{\beta}(u)$ be parameterized curves with $\vec{\beta}(u) \neq 0$ for all u. Now define the parameterized surface:

$$\vec{x}(u,v) = \vec{\alpha}(u) + v\,\vec{\beta}(u)$$

this is a v-**ruled surface** with **rulings** $\vec{\beta}(u)$ and **directrix** $\vec{\alpha}(u)$. A similar expression holds for u-ruled surfaces.

First Fundamental Form

The **First Fundamental Form** is:

$$I_P = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \vec{x}_u \cdot \vec{x}_v & \vec{x}_v \cdot \vec{x}_v \end{pmatrix}$$

Thm. if for each $P \in M$ there exists $\vec{x}: U \to M$ with $\vec{x}(u_0, v_0) = P$ and $\vec{x}^*: U \to M^*$ with $I_P = I_P^*$, then M and M^* are **locally isometric**.
Def. a surface is **conformal** if for all $P \in M$, $E = G$ and $F = 0$. Two surfaces are conformal if $I_P = \lambda I_P^*$ for some scalar-valued function $\lambda(u,v)$.
The **surface area** is given by:

$$A = \int_U du dv\, ||\vec{x}_u \times \vec{x}_v|| = \int_U du dv\, \sqrt{EG - F^2}$$

Gauss Map

The **Gauss map** is $\vec{n}: M \to \Sigma$, where Σ is the unit sphere. For curves there are tangent, normal, and binormal spherical images $\vec{\alpha} \to \Sigma$.

Second Fundamental Form

The **Second Fundamental Form** is:

$$\mathit{II}_P = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \begin{pmatrix} \vec{x}_{uu} \cdot \vec{n} & \vec{x}_{uv} \cdot \vec{n} \\ \vec{x}_{uv} \cdot \vec{n} & \vec{x}_{vv} \cdot \vec{n} \end{pmatrix}$$

Meusnier's formula, if $\vec{\alpha}'(0) = \vec{T}$:

$$\kappa_n = (\mathit{II}_P \vec{T}) \cdot \vec{T} = \kappa \cos(\phi) = \kappa \vec{N} \cdot \vec{n}$$

where ϕ is the angle between the curve normal $\vec{N}$ and the surface normal $\vec{n}$.
Def. $\vec{V}$ is an **asymptotic direction** at P if $(\mathit{II}_P \vec{V}) \cdot \vec{V} = 0$.
Def. $\vec{\alpha}$ is an **asymptotic curve** if $(\mathit{II}_P \vec{T}) \cdot \vec{T} = 0$ for all $P \in \vec{\alpha}$.

The Shape Operator

The **shape operator** in matrix form:

$$S_P = I_P^{-1} \mathit{II}_P$$

the eigenvalues of S are the principal curvatures k_1 and k_2, the eigenvectors are the principal directions at point P.
Def. $\vec{\alpha}$ is a **line of curvature** if its tangent is always a principal direction.
Thm. If there exists an asymptotic direction at P, then $k_1 k_2 \leq 0$.
Thm. M is a subset of a plane if $S_P = 0$ for all $P \in M$.

Surface Curvature

The **Gaussian curvature** is:

$$K = \det(S_P) = k_1 k_2$$

The **mean curvature** is:

$$H = \tfrac{1}{2}\mathrm{tr}(S_P) = \tfrac{1}{2}(k_1 + k_2)$$

A surface is **flat** if $K = 0$, and a surface is **minimal** if $H = 0$.

Point	Requirement
Umbillic	$k_1 = k_2$
Planar	$k_1 = k_2 = 0$
Parabolic	$K = 0$; non-planar
Elliptic	$K > 0$
Hyperbolic	$K < 0$

Christoffel Symbols

Let the **Christoffel symbols** Γ_{ij}^k be:

$$\vec{x}_{uu} = \Gamma_{uu}^u \vec{x}_u + \Gamma_{uu}^v \vec{x}_v + \ell \vec{n}$$
$$\vec{x}_{uv} = \Gamma_{uv}^u \vec{x}_u + \Gamma_{uv}^v \vec{x}_v + \ell \vec{n}$$
$$\vec{x}_{vv} = \Gamma_{vv}^u \vec{x}_u + \Gamma_{vv}^v \vec{x}_v + \ell \vec{n}$$

In terms of the first-fundamental form:

$$\begin{pmatrix} \Gamma_{uu}^u \\ \Gamma_{uu}^v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_u/2 \\ F_u - E_v/2 \end{pmatrix}$$
$$\begin{pmatrix} \Gamma_{uv}^u \\ \Gamma_{uv}^v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_v/2 \\ G_u/2 \end{pmatrix}$$
$$\begin{pmatrix} \Gamma_{vv}^u \\ \Gamma_{vv}^v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} F_v - G_u/2 \\ G_v/2 \end{pmatrix}$$

Codazzi & Gauss Equations

The **Codazzi equations** are:

$$\ell_v - m_u = \ell \Gamma_{uv}^u + m(\Gamma_{uv}^v - \Gamma_{uu}^u) - n\Gamma_{uu}^v$$
$$m_v - n_u = \ell \Gamma_{vv}^u + m(\Gamma_{vv}^v - \Gamma_{uv}^u) - n\Gamma_{uv}^v$$

The **Gauss equations** are:

$$EK = (\Gamma_{uu}^v)_v + (\Gamma_{uv}^v)_u + \Gamma_{uu}^u \Gamma_{uv}^v + \Gamma_{uu}^v \Gamma_{vv}^v - \Gamma_{uv}^u \Gamma_{uu}^v - (\Gamma_{uv}^v)^2$$
$$FK = (\Gamma_{uv}^u)_u - (\Gamma_{uu}^u)_v + \Gamma_{uv}^v \Gamma_{uv}^u - \Gamma_{uu}^v \Gamma_{vv}^u$$
$$FK = (\Gamma_{uv}^v)_v - (\Gamma_{vv}^v)_u + \Gamma_{uv}^u \Gamma_{uv}^v - \Gamma_{vv}^u \Gamma_{uu}^v$$
$$GK = (\Gamma_{vv}^u)_u - (\Gamma_{uv}^u)_v + \Gamma_{vv}^u \Gamma_{uu}^u + \Gamma_{vv}^v \Gamma_{uv}^u - (\Gamma_{uv}^u)^2 - \Gamma_{uv}^v \Gamma_{vv}^u$$

Thm. If two surfaces are locally isometric, their Gaussian curvatures at corresponding points are equal.
Thm. Suppose $M \subset \mathbb{R}^3$ is a compact surface. Then there is a point $P \in M$ with $K(P) > 0$.
Thm. If M is a smooth, compact surface of constant Gaussian curvature K, then $K > 0$ and M must be a sphere of radius $1/\sqrt{K}$.

Gauss's Theorema Egregium

Thm. The Gaussian curvature is determined by only the (partials of the) first fundamental form I.
We have in general that:

$$K = \frac{\ell n - m^2}{EG - F^2}$$

So, if $F = 0$, we see that:

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

Fundamental Theorem of Surfaces

Thm. Two surfaces $\vec{x}, \vec{x}^*: U \to \mathbb{R}^3$ are congruent iff $I = I^*$ and $\mathit{II} = \pm \mathit{II}^*$.
Thm. Given differentiable functions E, F, G, ℓ, m, n with $E > 0$ and $EG - F^2 > 0$ that satisfy the Codazzi and Gauss equations, then there exists a parameterized surface $\vec{x}(u,v)$ with the respective I and II.

Geodesics

Def. A **geodesic** is a curve, $\vec{\alpha}(t) = \vec{x}(u(t), v(t))$ on M that satisfies:

$$u'' + \Gamma_{uu}^u (u')^2 + 2\Gamma_{uv}^u u'v' + \Gamma_{vv}^u (v')^2 = 0$$
$$v'' + \Gamma_{uu}^v (u')^2 + 2\Gamma_{uv}^v u'v' + \Gamma_{vv}^v (v')^2 = 0$$