Let us consider the two-point correlation function

$$iG^{\alpha\beta}(t,t') = \langle a^\alpha(t)\bar{a}^\beta(t')\rangle = \int \mathcal{D}[a^c, a^q] a^\alpha(t)\bar{a}^\beta(t') e^{iS[a^c,a^q]} \tag{1}$$

where α, β can assume values of either c, q. Now we are given that

$$iS[a^+, a^-] = i\int_{-\infty}^{\infty} dt\ [\bar{a}^+ i\partial_t a^+ - H(\bar{a}^+, a^+) - \bar{a}^- i\partial_t a^- + H(\bar{a}^-, a^-)] \tag{2}$$

where the second-quantized (normal-ordered) Hamiltonian is

$$\hat{H} = \omega_0 \hat{a}^\dagger \hat{a} + \frac{K}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \tag{3}$$

We can now Keldysh rotate

$$a^+ = \frac{a_c + a_q}{\sqrt{2}}, \qquad a^- = \frac{a_c - a_q}{\sqrt{2}}, \qquad \bar{a}^+ = \frac{\bar{a}_c + \bar{a}_q}{\sqrt{2}}, \qquad \bar{a}^- = \frac{\bar{a}_c - \bar{a}_q}{\sqrt{2}} \tag{4}$$

to find

$$iS[a^c, a^q] = iS_0 + iS_{\text{int}} = i\int_{-\infty}^{\infty} dt\ \bar{a}^c(i\partial_t - \omega_0)a^q + \bar{a}^q(i\partial_t - \omega_0)a^c \tag{5}$$

$$-i\int_{-\infty}^{\infty} dt\ \frac{K}{2}(\bar{a}^c a^q + \bar{a}^q a^c)(\bar{a}^c a^c + \bar{a}^q a^q) \tag{6}$$

where S_0 is quadratic and S_{int} is quartic. We have

$$Z = e^{iS} = e^{iS_0 + iS_{\text{int}}} = e^{iS_0}e^{iS_{\text{int}}} \approx e^{iS_0}(1 + iS_{\text{int}} - S_{\text{int}}^2) \tag{7}$$

where we used that S are scalars so there is no need to use the BCH formula. Now for bosonic (coherent states/complex scalar a), weighted by $e^{i\sum_{ij}\bar{a}_i G^{-1}_{ij} a_j} = e^{-\sum_{ij}\bar{a}_i (iG_{ij})^{-1} a_j}$ (where $(iG_{ij})^{-1}$ is invertible and has non-negative real parts of all eigenvalues) the Wick theorem is

$$\langle a_i \bar{a}_j\rangle = iG_{ij} \tag{8}$$

$$\langle a_i a_j \bar{a}_k \bar{a}_l\rangle = iG_{ik}iG_{jl} + iG_{il}iG_{jk} \tag{9}$$

$$\langle a_i a_j a_k \bar{a}_l \bar{a}_m \bar{a}_n\rangle = iG_{il}iG_{jm}iG_{kn} + iG_{il}iG_{jn}iG_{km} + iG_{im}iG_{jl}iG_{kn} + iG_{im}iG_{jn}iG_{kl} + iG_{in}iG_{jm}iG_{kl} + iG_{in}iG_{jl}iG_{kn} \tag{10}$$

Now expressing as a matrix and inserting an infinitesimal damping η

$$iS_0 = i\int_{-\infty}^{\infty} dt\ \begin{pmatrix}\bar{a}^c \\ \bar{a}^q\end{pmatrix} \begin{pmatrix} 0 & i\partial_t - \omega_0 \\ i\partial_t - \omega_0 & 2i\eta\end{pmatrix} \begin{pmatrix} a^c \\ a^q\end{pmatrix} \tag{11}$$

so we see that

$$G = \begin{pmatrix} 0 & i\partial_t - \omega_0 \\ i\partial_t - \omega_0 & 2i\eta\end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{i\partial_t - \omega_0}2i\eta\frac{1}{i\partial_t-\omega_0} & \frac{1}{i\partial_t - \omega_0} \\ \frac{1}{i\partial_t-\omega_0} & 0\end{pmatrix} = \begin{pmatrix} G^K & G^R \\ G^A & 0\end{pmatrix} \tag{12}$$

where $G_{qq} = 0$.

Now we have (where we indicate the time dependences with subscripts $a(t_1) \to\ a_1$ and $a(t_2) \to\ a_2$)

$$Z \approx \int \mathcal{D}[\bar{a}, a](1 - i\frac{K}{2}(\bar{a}^c a^q + \bar{a}^q a^c)_1(\bar{a}^c a^c + \bar{a}^q a^q)_1 \tag{13}$$

$$- \frac{K^2}{4}(\bar{a}^c a^q + \bar{a}^q a^c)_1(\bar{a}^c a^c + \bar{a}^q a^q)_1(\bar{a}^c a^q + \bar{a}^q a^c)_2(\bar{a}^c a^c + \bar{a}^q a^q)_2)e^{iS_0} \tag{14}$$

For the linear in K terms we have that (where we are free to commute variables since they are scalars, and everything here is at time t_1)

$$\langle \bar{a}^c a^q \bar{a}^c a^c \rangle = \langle a^q a^c \bar{a}^c \bar{a}^c \rangle = -G_{qc}G_{cc} - G_{qc}G_{cc} = -2G^A G^K \tag{15}$$

$$\langle \bar{a}^c a^q \bar{a}^q a^q \rangle = \langle a^q a^q \bar{a}^c \bar{a}^q \rangle = -G_{qc}G_{qq} - G_{qq}G_{qc} = 0 \tag{16}$$

$$\langle \bar{a}^q a^c \bar{a}^c a^c \rangle = \langle a^c a^c \bar{a}^q \bar{a}^c \rangle = -G_{cq}G_{cc} - G_{cc}G_{cq} = -G^R G^K - G^K G^R = -2G^R G^K \tag{17}$$

$$\langle \bar{a}^q a^c \bar{a}^q a^q \rangle = \langle a^c a^q \bar{a}^q \bar{a}^q \rangle = -G_{cq}G_{qq} - G_{cq}G_{qq} = 0 \tag{18}$$

Now we recall the relation $G^R + G^A = 0$ (and $G^R - G^A = -i$) so that

$$\langle \bar{a}^c a^q \bar{a}^c a^c \rangle + \langle \bar{a}^c a^q \bar{a}^q a^q \rangle + \langle \bar{a}^q a^c \bar{a}^c a^c \rangle + \langle \bar{a}^q a^c \bar{a}^q a^q \rangle = -2(G^A + G^R)G^K = 0 \tag{19}$$

where the zero only holds after the contour integral (see Kamenev Eq. 2.45). The sixteen 8-point functions may be evaluated similarly where the substitution $G^R(t_i, t_i) = 0$ and $G^A(t_i, t_i) = 0$ is justified by the discussion under Eq. 2.45. The terms that remain after doing this then vanish once $G^R(t, t')G^R(t', t) = 0$, $G^A(t, t')G^A(t', t) = 0$, $G^R(t, t')G^A(t, t') = 0$ are taken into account (which follow from the step functions).