Motivation and Introduction

In these notes we develop the Keldysh action for both bosonic and fermionic systems, as well as those governed by dissipative dynamics of the Lindblad master equation. We start by introducing the Keldysh contour, consider one-state systems, discretize time, obtain the Green's functions, and take the continuum limit. We then generalize to more degrees of freedom, and consider the generating functional created with the Keldysh action and source fields. Finally we derive the Keldysh action for systems governed by the Lindblad master equation (instead of the von Neumann master equation).

These notes largely follow Section 2 and 5 of Alex Kamenev and Alex Levchenko, arXiv:0901.3586, and Section IIA of L. M. Sieberer, M. Buchhold, and S. Diehl arXiv:1512.00637.

Expectation Values

We want expectation values

$$\langle X(t)\rangle = \frac{\mathrm{Tr}(X\rho(t))}{\mathrm{Tr}(\rho(t))} = \left.\frac{\delta Z}{\delta\alpha}\right|_{\alpha=0}$$

Where $Z = e^{iS+\alpha X}$ is the generating functional that will become important later. Now, we have the differential equation

$$\dot\rho = -i[H,\rho]$$

subject to $\rho(-\infty) =$ some known state. Now the solution to this is the rotation from $-\infty$ to t

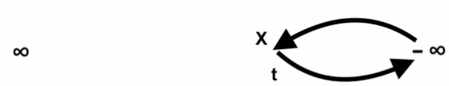
$$\rho(t) = U_{t\leftarrow-\infty}\rho(-\infty)U^\dagger_{t\leftarrow-\infty} = U_{t\leftarrow-\infty}\rho(-\infty)U_{-\infty\leftarrow t}$$

Where the unitary rotation matrix is given by

$$U_{t_f\leftarrow t_i} = \mathcal{T}e^{-i\int_{t_i}^{t_f}d\bar t\; H(\bar t)} = \lim_{N\to\infty}\prod_{n=1}^{N} e^{-iH(n\tau)\tau}$$

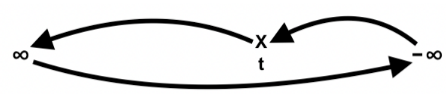
where we used the Suzuki-Trotter expansion, and $\tau = (t_f - t_i)/N$. Continuing in this vein and using the cyclicity of the trace we have

$$\langle X(t)\rangle = \frac{\mathrm{Tr}(U_{-\infty\leftarrow t}XU_{t\leftarrow-\infty}\rho(-\infty))}{\mathrm{Tr}(\rho(t))} =$$



where instead we rotate X from t to $-\infty$. We can make this more symmetric by rotating out to $+\infty$ as well

$$\langle X(t)\rangle = \frac{\mathrm{Tr}(U_{-\infty\leftarrow\infty}U_{\infty\leftarrow t}XU_{t\leftarrow-\infty}\rho(-\infty))}{\mathrm{Tr}(\rho(t))} =$$



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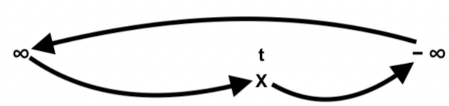

where we used $\mathbb{1} = U_{-\infty\leftarrow\infty}U_{\infty\leftarrow-\infty}$ and noted that we can hit the operator either on the out contour or the back contour.

Coherent State Calculations

Fermions

Consider a single fermionic mode with energy ϵ_0

$$H = \epsilon_0 c^\dagger c$$

The partition function is then

$$Z = \frac{\mathrm{Tr}(U_{\text{contour}}\rho)}{\mathrm{Tr}(\rho)}$$

Now, let us reformulate this partition function in terms of quantities we are more familiar with in field theory. We have

$$\mathbb{1} = \int \frac{d\bar{\psi}_n\, d\psi_n}{\pi} e^{-\bar{\psi}_n \psi_n} |\psi_n\rangle\langle\psi_n|$$

We can now bring back the Suzuki-Trotter expansion of the U_{contour} and insert this between each time slice. This is useful since

$$\begin{aligned}\langle\psi_{n+1}|U_{t_{n+1}\leftarrow t_n}|\psi_n\rangle &\approx \text{integral as finite difference}\\ &= \text{Taylor expand to 1st order}\\ &= \text{use coherent state identity}\\ &= \text{Taylor unexpand}\\ &= e^{\bar{\psi}_{n+1}\psi_n} e^{-iH(\bar{\psi}_{n+1},\psi_n)\tau}\end{aligned}$$

for $\tau = t_{n+1} - t_n$, where without loss of generality we took the Hamiltonian to be normal ordered.

For now let $U_{\text{contour}} = U_{-\infty\leftarrow\infty} U_{\infty\leftarrow-\infty}$ and break this into $2N$ steps. Then

$$Z = \frac{1}{\mathrm{Tr}(\rho(-\infty))} \int \prod_{n=1}^{2N} \frac{d\bar{\psi}_n\ d\psi_n}{\pi} e^{i\sum_{n,n'}^{2N} \bar{\psi}_n G^{-1}_{nn'} \psi_{n'}}$$

Where the inverse Green's function matrix is

$$iG^{-1}_{nn'} = \begin{pmatrix} -1 & & & & & -\rho(\epsilon_0) \\ 1-h & -1 & & & & \\ & 1-h & -1 & & & \\ & & 1 & -1 & & \\ & & & 1+h & -1 & \\ & & & & 1+h & -1 \end{pmatrix}$$

where $h = i\epsilon_0(\infty - -\infty)/N$.

Bosons

Consider a single bosonic mode with energy ω_0

$$H = \omega_0 b^\dagger b$$

The partition function is then

$$Z = \frac{\mathrm{Tr}(U_{\text{contour}}\rho)}{\mathrm{Tr}(\rho)}$$

Now, let us reformulate this partition function in terms of quantities we are more familiar with in field theory. We have

$$\mathbb{1} = \int \frac{d\,\mathrm{Re}\,\varphi_n\ d\,\mathrm{Im}\,\varphi_n}{\pi} e^{-|\varphi_n|^2} |\varphi_n\rangle\langle\varphi_n|$$

We can now bring back the Suzuki-Trotter expansion of the U_{contour} and insert this between each time slice. This is useful since

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Continuum Actions

We can formulate this as an action, $Z = e^{iS}$

$$\begin{aligned} S = &\sum_{n,n' \in \text{contour}} \bar{\psi}_n G^{-1}_{nn'} \psi_{n'} \\ =& i\bar{\psi}_1(\psi_1 + \rho(\epsilon_0)\psi_{2N}) \\ &+ \sum_{n=2}^{2N} \left(i\bar{\psi}_n \frac{\psi_n - \psi_{n-1}}{\tau_n} - \epsilon_0 \bar{\psi}_n \psi_{n-1} \right) \tau_n \end{aligned}$$

Now take the continuum limit $N \to \infty$, so that $\psi_n \mapsto \psi(t)$, and drop the boundary terms. Breaking into "left" and "right" fields we find

$$\begin{aligned} S &= \int_{\text{contour}} dt\ \bar{\psi}(t)(i\partial_t - \epsilon_0)\psi(t) \\ &= \int_{-\infty}^{\infty} dt\ \bar{\psi}_+(i\partial_t - \epsilon_0)\psi_+ - \bar{\psi}_-(i\partial_t - \epsilon_0)\psi_- \end{aligned}$$

that appear decoupled in the continuum limit, but are coupled by the off-diagonal terms of $G^{-1}_{nn'}$.

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Green's Functions

Now the Green's function matrix is, with $\bar{\rho} \equiv -\rho$

$$iG^{-1}_{nn'} = \frac{1}{1-\bar{\rho}} \begin{pmatrix} 1 & \bar{\rho}e^h & \bar{\rho}e^{2h} & \bar{\rho}e^{2h} & \bar{\rho}e^h & \bar{\rho} \\ e^{-h} & 1 & \bar{\rho}e^h & \bar{\rho}e^h & \bar{\rho} & \bar{\rho}e^{-h} \\ e^{-2h} & e^{-h} & 1 & \bar{\rho} & \bar{\rho}e^{-h} & \bar{\rho}e^{-2h} \\ e^{-2h} & e^{-h} & 1 & 1 & \bar{\rho}e^{-h} & \bar{\rho}e^{-2h} \\ e^{-h} & 1 & e^h & e^h & 1 & \bar{\rho}e^{-h} \\ 1 & e^h & e^{2h} & e^{2h} & e^h & 1 \end{pmatrix}$$

Taking the Larkin-Ovchinnikov transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \\ \bar{\psi}_+ \\ \bar{\psi}_- \end{pmatrix}$$

Which leads to a nice form of the Green's function matrix, where a, b run over $1, 2$

$$G_{ab}(t,t') = -i\langle \psi_a(t)\bar{\psi}_b(t')\rangle = \begin{pmatrix} G^R(t,t') & G^K(t,t') \\ 0 & G^A(t,t') \end{pmatrix}$$

Explicitly

$$\begin{aligned} G^{R/A}(t,t') &= \mp i\theta(t-t')e^{-i\epsilon_0(t-t')} \\ G^K(t,t') &= -i(1-2n_F)e^{-i\epsilon_0(t-t')} \end{aligned}$$

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$$iG^{-1}_{nn'} = \frac{1}{1-\rho} \begin{pmatrix} 1 & \rho e^h & \rho e^{2h} & \rho e^{2h} & \rho e^h & \rho \\ e^{-h} & 1 & \rho e^h & \rho e^h & \rho & \rho e^{-h} \\ e^{-2h} & e^{-h} & 1 & \rho & \rho e^{-h} & \rho e^{-2h} \\ e^{-2h} & e^{-h} & 1 & 1 & \rho e^{-h} & \rho e^{-2h} \\ e^{-h} & 1 & e^h & e^h & 1 & \rho e^{-h} \\ 1 & e^h & e^{2h} & e^{2h} & e^h & 1 \end{pmatrix}$$

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$$\begin{aligned} G^{R/A}(t,t') &= \mp i\theta(t-t')e^{-i\omega_0(t-t')} \\ G^K(t,t') &= -i(1+2n_B)e^{-i\omega_0(t-t')} \end{aligned}$$

Generalization to More DOFs

The generalization to more degrees of freedom is straightforward and involves considering matrices with blocks rather than scalars. In particular, if we expand to a continuous index $\boldsymbol{k}$ and a discrete index n then we have

$$\begin{aligned}
S &= \int_{-\infty}^{\infty} dt \int_{BZ} d\boldsymbol{k} \sum_n \bar{\psi}_+ i\partial_t \psi_+ - \bar{\psi}_- i\partial_t \psi_- - \bar{\psi}_+ H \psi_+ + \bar{\psi}_- H \psi_- \\
&= \int_{-\infty}^{\infty} dt \int_{BZ} d\boldsymbol{k} \sum_n \bar{\psi}_+ i\partial_t \psi_+ - \bar{\psi}_- i\partial_t \psi_- - i(\bar{\psi}_+(-iH)\psi_+ + \bar{\psi}_-(iH)\psi_-)
\end{aligned}$$

Where this expression is the same for fermions and bosons (since we haven't rotated yet), although we have written it suggestively for (spinful or spinless) electronic band physics. Now we recall that

$$\dot{\rho} = -i[H,\rho] = -iH\rho + i\rho H$$

So we see something essential about the Keldysh contour: operators acting on the left reside on the $+$ contour, while operators acting on the right reside on the $-$ contour. Fundamentally for mixed states, time evolution involves a commutator so there will be (time-evolution) operators on both branches, while for pure states only one operator is needed. This also suggests that in the Lindblad formalism where operators act on *both* the left and right there will be terms of the sort $\bar{\psi}_\pm X \psi_\mp$ which explicitly break the unitary time evolution.

Keldysh Action for Lindblad Systems

Now, if we have

$$\dot{\rho} = -i\mathcal{L}[\rho]$$

Where for the Lindblad master equation is

$$\mathcal{L} = [H,\rho] - i\frac{\Gamma}{2}\sum_m (\{J_m^\dagger J_m, \rho\} - 2J_m \rho J_m^\dagger)$$

Then

$$S = \int_{-\infty}^{\infty} dt \int_{\text{dof}} \bar{\psi}_+ i\partial_t \psi_+ - \bar{\psi}_- i\partial_t \psi_- - \langle\langle \mathcal{L} \rangle\rangle$$

Where we mean

$$\begin{aligned}
\langle\langle \mathcal{L} \rangle\rangle_{\text{bosons}} &= \bar{\psi}_+ H \psi_+ - \bar{\psi}_- H \psi_- - i\frac{\Gamma}{2}\sum_m \bar{\psi}_+ J_m^\dagger J_m \psi_+ + \bar{\psi}_- J_m^\dagger J_m \psi_- - 2\bar{\psi}_+ J_m \psi_+ \bar{\psi}_- J_m^\dagger \psi_-) \\
\langle\langle \mathcal{L} \rangle\rangle_{\text{fermions}} &= \bar{\psi}_+ H \psi_+ - \bar{\psi}_- H \psi_- - i\frac{\Gamma}{2}\sum_m \bar{\psi}_+ J_m^\dagger J_m \psi_+ + \bar{\psi}_- J_m^\dagger J_m \psi_- - 2\bar{\psi}_- J_m^\dagger \psi_- \bar{\psi}_+ J_m \psi_+)
\end{aligned}$$

This follows from a similar derivation to the one given above where the only generator of evolution is $[H,\rho]$. The swap in ordering between bosons and fermions comes from the fact that fermionic coherent states change sign on the two contours while bosonic coherent states don't change sign.

Fermionic Response Functions

Take a potential/source added to the action

$$S \mapsto S + S'$$

The partition function then becomes a generating function

$$Z = \langle e^{iS'} \rangle$$

For example one that couples linearly to the density

$$\begin{aligned} S' &= \int_{\text{contour}} dt \; V(t)\bar{\psi}(t)\psi(t) \\ &= \int_{-\infty}^{\infty} dt \; V_+\bar{\psi}_+\psi_+ - V_-\bar{\psi}_-\psi_- \\ &= \int_{-\infty}^{\infty} dt \; V^1(\bar{\psi}_+\psi_+ - \bar{\psi}_-\psi_-) + V^2(\bar{\psi}_+\psi_+ + \bar{\psi}_-\psi_-) \\ &= \int_{-\infty}^{\infty} dt \; V^1(\bar{\psi}_1\psi_1 + \bar{\psi}_2\bar{\psi}_2) + V^2(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1) \\ &= \int_{-\infty}^{\infty} dt \; V^1\bar{\psi}_a\sigma_0^{ab}\psi_b + V^2\bar{\psi}_a\sigma_1^{ab}\psi_b \\ &= \text{Tr}(\bar{\Psi}V\Psi) \end{aligned}$$

where we use Einstein summation notation and σ_0 and σ_1 are the Pauli matrices which simplify notation. $\Psi = (\psi_1, \psi_2)$ and $V = V^1\sigma_0 + V^2\sigma_1$ are a vector and matrix that encapsulate this. Note that V^2 couples to the physical density. This leads us to

$$\boldsymbol{\rho}(x) = -\frac{i}{2}\frac{\delta Z}{\delta V_2(x)}\bigg|_{V_2=0}$$

Whence the susceptibility is

$$\Pi^R(x,x') = \frac{\delta\boldsymbol{\rho}(x)}{\delta V_1(x')}\bigg|_{V_1=0}$$

More generically

$$\Pi^{ab} = -\frac{i}{2}\frac{\delta^2 Z}{\delta V^b(x')\delta V^a(x)}\bigg|_{V_a=0}\bigg|_{V_b=0} = -\frac{i}{2}\frac{\delta^2 \ln(Z)}{\delta V^b(x')\delta V^a(x)}\bigg|_{V_a=0}\bigg|_{V_b=0} = \begin{pmatrix} 0 & \Pi^A(x,x') \\ \Pi^R(x,x') & \Pi^K(x,x') \end{pmatrix}$$

where the seemingly strange second equality holds because of the normalization of the generating function. Next, we consider the generating function

$$Z = e^{iS+iS'} = \int D[\bar{\Psi},\Psi] \; e^{i\int dr \;\; \bar{\Psi}(G^{-1}+V^a\sigma^{a-1})\Psi} = \det(1 + GV^a\sigma^{a-1}) = e^{\text{Tr}(\ln(1+GV^a\sigma^{a-1}))}$$

where we evaluated the Gaussian integral and used that $\det(M) = e^{\text{Tr}\ln(M)}$. Expanding this, we find the standard one-bubble diagram (where the Kubo formula is the Π^{21} component).

$$\Pi^{ab} = -\frac{i}{2}\text{Tr}(\sigma^{a-1}G(x,x')\sigma^{b-1}G(x',x))$$

Which provides a way to calculate response for out of equilibrium systems using out of equilibrium Green's functions.