Complex Numbers

i is the imaginary unit which satisfies $i^2 = -1$. In terms of real numbers x and y, the complex number z is:

$$z = x + iy$$

Its complex conjugate is given by:

$$z^* = x - iy$$

The magnitude of a complex number z:

$$r = |z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

Complex numbers can be expressed in polar form with phases $\theta = \arg(z)$:

$$z = re^{i\theta}$$

To go from polar form to Cartesian, it helps to use Euler's formula:

$$e^z = e^x\big(\cos(y) + i\sin(y)\big)$$

Some identities that may help are:

$$\frac{1}{z} = \frac{z^*}{|z|^2}, \quad x = \frac{z + z^*}{2}, \quad y = \frac{z - z^*}{2i}$$

The square root of a complex number:

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}$$

The Residue Theorem

A function f is analytic at z if its power series converges at z. Let a function $f(z)$ be expressed by its power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

The "residue" of $f(z)$ at z_0 is

$$\text{Res}[f(z), z_0] = a_{-1}$$

If $f(z)$ is non-zero and analytic at z_0:

$$\text{Res}\left[\frac{f(z)}{(z - z_0)^n}, z_0\right] = \frac{f^{(n-1)}(z_0)}{(n-1)!}$$

If $f(z)$ and $g(z)$ are analytic at z_0 and $g'(z_0) \neq 0$ then:

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z)}{g'(z)}$$

Residue Theorem: for any (counter-clockwise) closed loop γ in which there are a set of isolated singularities $\{z_j\}$:

$$\oint_\gamma f(z)dz = 2\pi i \sum_j \text{Res}[f(z), z_j]$$

If P and Q are polynomials, $k > 0$ is real, and $\deg(Q) \geq \deg(P) + 2$, then:

$$\int_{-\infty}^{\infty} dx\, \frac{P(x)}{Q(x)} = 2\pi i \sum_j \text{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$$

$$\int_{-\infty}^{\infty} dx\, \frac{P(x)}{Q(x)} e^{ikx} = \frac{2\pi i}{e^k} \sum_j \text{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$$

Linear Functions

Linear functions respect addition and scalar multiplication

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(cf)(x) = c\, f(x)$$

A "linear combination" of functions is

$$\sum\nolimits_n a_n f_n(x)$$

where a_n are scalars.

A set $\{f_n\}$ is "linearly independent" if

$$\sum\nolimits_n a_n f_n(x) = 0$$

iff $a_n = 0$ for all $n = 1, \ldots, N$.

A set $\{f_n\}$ is "complete" if for every g

$$g(x) = \sum\nolimits_n a_n f_n(x)$$

for some scalars a_n.

Functions f, g are "orthogonal" on D if

$$\langle f, g\rangle_D = \int_D dx\ f(x)g(x) = 0$$

where $\langle f, g\rangle_D$ is an inner product on D.

A function f is "normalized" on D if

$$\langle f, f\rangle_D = \int_D dx\ f(x)f(x) = 1$$

Linear Transformations

$L : F \to G$ is a linear transformation iff

$$L(af_1 + bf_2) = a\, L(f_1) + b\, L(f_2)$$

for all scalars a, b and functions $f_1, f_2 \in F$. Note: differentiation and integration are linear transformations.

The matrix element of L with f_1, f_2 is

$$\langle f_1, Lf_2\rangle_D = \int_D dx\ f_1(x)\big(Lf_2(x)\big)$$

Under a change of bases $\{f_n\} \mapsto \{g_n\}$, the transformation is the identity whose matrix elements are given by

$$\langle g_m, \mathbf{1} f_n\rangle_D = \int_D dx\ g_m(x) f_n(x)$$

If for some scalar λ and function u

$$Lu = \lambda u$$

then λ is an eigenvalue of L and u is the corresponding eigenfunction.

Eigenvalues are often found by solving

$$\det(L - \lambda I) = 0$$

where L is expressed in some basis.

The adjoint $L^\dagger$ of a linear transformation L is the $L^\dagger$ that for all f_1, f_2 fulfills

$$\langle f_1, Lf_2\rangle_D = \langle L^\dagger f_1, f_2\rangle_D$$

If $L = L^\dagger$, then L is self-adjoint, or "Hermitian" and has real eigenvalues.

Characteristic Equations

Consider the differential equation

$$Lu(x) = 0$$

and suppose that

$$L = \sum_{n=0}^{N} a_n \frac{d^n}{dx^n}$$

Then since scalar multiples of the identity commute with every operator

$$L = \prod_{m=0}^{N} \frac{d}{dx} - r_m \mathbf{1}$$

where r_m are the roots of the equation

$$\sum_{n=0}^{N} a_n x^n = 0$$

We then have $N + 1$ decoupled first-order equations whose solution is

$$u(x) = \sum_{m=0}^{N} c_m e^{r_m x}$$

for some scalars c_m.

Example: Taylor Series

The Taylor expansion of f about x_0 is

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}$$

which is a linear combination of powers of x (which are linearly independent). The first terms of a Taylor expansion are a good approximation to f if $x \sim x_0$, provided that f is a nice function.

The Power Series Method

Consider the differential equation

$$Lu(x) = 0$$

and suppose that

$$L = \sum_{n=0}^{N} \Big(\sum_{m=0}^{M} a_{m,n} x^m\Big) \frac{d^n}{dx^n}$$

Then suppose the ansatz

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

and write down $Lu(x)$ and reindex so that all sums go over the same n—since different powers are linearly independent, the coefficients are each zero, which gives a recursion relation. Often these recursion relations can be solved for some known (special) function with only the first terms undetermined.

Fourier Series

It happens to be the case that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}$ where n ranges over the natural numbers are a complete and orthogonal set of functions for all L-periodic functions. With normalization $1/\sqrt{L}$ we define

$$s_n = \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi}{L}x\right), \ \ c_n = \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi}{L}x\right)$$

from which we can decompose any L-periodic function as

$$f(x) = \sum_{n=0}^{\infty} \langle s_n, f\rangle_{[0,2L]} s_n + \langle c_n, f\rangle_{[0,2L]} c_n$$

where

$$\langle a, b\rangle_{[0,2L]} = \int_0^{2L} dx \ a(x)b(x)$$

is a scalar, the inner product on $[0, 2L]$.

Orthogonormality means that

$$\langle s_n, s_m\rangle = \langle c_n, c_m\rangle = \delta_{nm}$$

and $\langle s_n, c_m\rangle = 0$.

Sturm-Liouville Theory

Sines and cosines are not the only complete orthogonal functions, but they are good for the wave equation on rectangular geometries. For different differential equations on different geometries, different orthogonal functions may be preferable. The expansion then works the exact same way.

Let $p(x), r(x) \geq 0$ and p, q, r continuous with continuous derivatives, then

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + q(x)u = -\lambda r(x)u$$

is a Sturm-Liouville equation.

If we have the boundary conditions:

$$\alpha u(0,t) + \beta u'(0,t) = 0$$
$$\gamma u(L,t) + \delta u'(L,t) = 0$$

then the Sturm-Liouville operator is self-adjoint, and so the λ are real.

If $q \leq 0$, $r > 0$, $p > 0$, $puu'|_0 \leq 0$, and $puu'|_L \leq 0$ then $\lambda \geq 0$ and we get a complete set of wave solutions $\{u_n\}$ where λ_n are increasing. The λ_n and thence the u_n can then be found using a shooting method (counting nodes).

Aside: this is equivalent to showing a Schrödinger equation has a ground state.

Fourier Transforms

If we don't have a L-periodic function we can still decompose it by integrating over all frequencies k instead of just $k = n\pi/L$. This representation is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \ \widetilde{f}(k)e^{+ikx}$$

where $\widetilde{f}(k)$ is given by

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} dx \ f(x)e^{-ikx}$$

This is useful because the Fourier transform turns calculus into algebra: d/dx transforms into $-ik$.

Laplace Transforms

A formulation that is useful for signals starting at $t = 0$ is the Laplace transform (the Wick rotation of the Fourier transform) of such a function $f(t)$

$$F(z) = \int_0^{\infty} dt \ f(t)e^{-zt}$$

whose inverse is

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \ F(z)e^{+zt}$$

with c to the right of all $F(z)$'s poles.

Dirac Delta Functions

A δ-function is any function where

$$\int_{-\infty}^{\infty} dx \ f(x)\delta(x) = f(0)$$

for all $f(x)$ which means that $\delta(x)$ has integral 1 and is peaked at 0.

If $g(x)$ has integral 1 on $\mathbb{R}$, then

$$g_\eta(x) = (1/\eta)g(x/\eta)$$

also has integral 1 on $\mathbb{R}$, and

$$\lim_{\eta\to 0} g_\eta(x) = \delta(x)$$

from which we can find as many representations of the δ-function as we want.

For example, the Poisson kernel:

$$\pi\delta(x) = \lim_{\eta\to 0}\frac{1}{\eta}\frac{1}{1+(x/\eta)^2}$$

from which one can show that

$$\lim_{\eta\to 0}\frac{1}{x+i\eta} = -i\pi\,\delta(x)$$

For a constant scalar c:

$$\delta(cx) = \delta(x)/|c|$$

and δ has units inverse to its argument.

For continuously differentiable $f(x)$

$$\delta(f(x)) = \sum_{\text{roots } i}\frac{\delta(x-x_i)}{|f'(x_i)|}$$

Green's Functions

Consider the differential equation

$$Lu(x) = f(x)$$

The Green's function of a linear operator is its *inverse* which means that

$$LG(x,x') = \delta(x-x')$$

The solution is then for domain D

$$u(x) = \int_D dx' \ G(x,x')f(x')$$

which follows directly from applying L.

So how do we find G?

1. Solve $LG = 0$ (for $x \neq x'$)
2. (Apply boundary conditions)
3. Exploit (1) continuity at x' and (2) $1 = \int_{x'-\epsilon}^{x'+\epsilon}\delta(x-x') = \int_{x'-\epsilon}^{x'+\epsilon} LG$ to match up the solutions at x'

Separation of Variables

This course only focuses on solving ODEs, and a prime way to get ODEs is from PDEs via separation of variables

$$Lf = 0$$

where a solution is postulated such as

$$f(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

separate ODEs are found and solved, and by uniqueness this is *the* solution.

For example with the wave equation

$$L = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

there are 11 separable coordinate systems (see Morse and Feshbach pg 655).

Aside: in spherical coordinates any Schrödinger equation with potential

$$V = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\varphi)}{r^2\sin^2(\theta)}$$

is separable (see Landau Vol 1 Sec 48).

Here Be Dragons

The methods above (and all known methods) fail if non-linearities such as a convective derivative are present in the differential equation. For example, (dis)proving that there are always smooth solutions to the Navier-Stokes equation will win you one million dollars. The simplified Euler equations have the convective derivative

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{1}{\rho}\nabla P$$

for velocity $\boldsymbol{v}$, density ρ, and pressure P.