Localization in Two-Dimensional Trivial and Chern Insulators

Transfer Matrix Methods

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Motivation

- Critical exponents describe system behavior near phase transitions (ex. conductivity)
- Metal-Insulator transitions in the presence of disorder lead to the Quantum Hall Effect in 2D
- The IQHE plateau transition critical exponent is 2.3 – 2.6, but the value is contested
- No analytic model $\rightarrow$ numerics
- Two approaches to answer this:
  1. Vary the models and parameters (ex. next nearest neighbor)
  2. Investigate larger systems

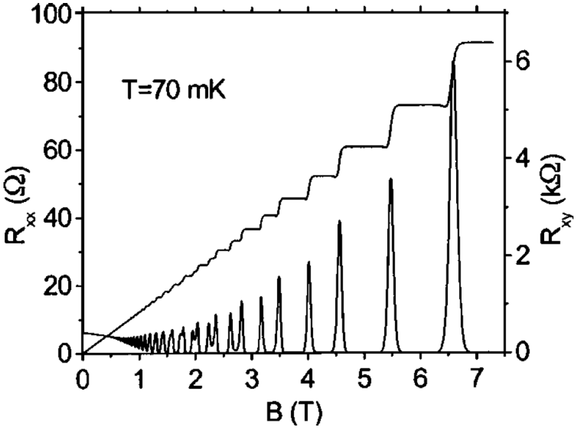


Figure: Transverse and Hall resistance in an InAs sample. From Zverev, et al, J. Appl. Phys. **96**, 6353 (2004).

2D Square Lattice

- Sites are $|\ell, w\rangle$
- Sites are orthogonal
- $|\psi\rangle = \sum_{\ell,w} C_{\ell,w}|\ell, w\rangle$
- ℓ ranges from 1 to L
- w ranges from 1 to W
- Cylindrical geometry
- Lattice constant is 1
- We consider $L \gg W$

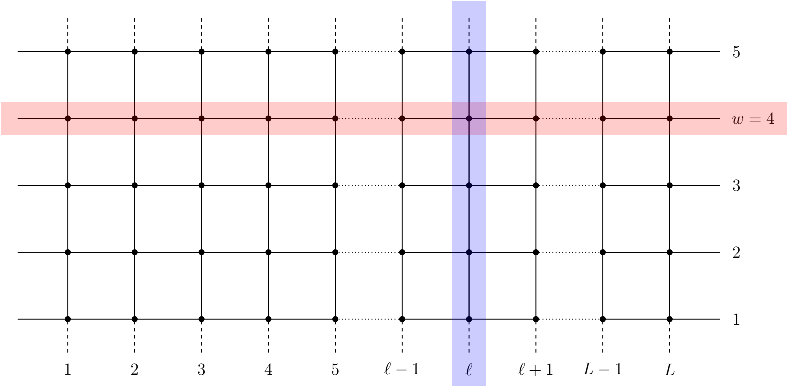


Figure: The 4th site in the ℓth cell, $|\ell, 4\rangle$, is highlighted

Tight-Binding Hamiltonians

General form:

$$H = T + V \implies H = \sum_{\text{sites}} (\text{onsite} + \text{transfer interactions})$$

Anderson Model: Anderson, Phys. Rev. **109**, 1492 (1958)

$$H = \sum_{\ell,w} \underbrace{\epsilon_{\ell,w} c^\dagger_{\ell,w} c_{\ell,w}}_{\text{onsite potential}} + \underbrace{t(c^\dagger_{\ell,w+1} + c^\dagger_{\ell,w-1} + c^\dagger_{\ell+1,w} + c^\dagger_{\ell-1,w}) c_{\ell,w}}_{\text{nearest-neighbor transfer}}$$

Hofstadter Model: Hofstadter, Phys. Rev. B **14**, 2239 (1976)

$$H = \sum_{\ell,w} \underbrace{\epsilon_{\ell,w} c^\dagger_{\ell,w} c_{\ell,w}}_{\text{onsite potential}} + \underbrace{t(c^\dagger_{\ell,w+1} + c^\dagger_{\ell,w-1}) c_{\ell,w}}_{\text{NN intra-cell transfer}} + \underbrace{t(e^{-i2\pi\alpha\ell} c^\dagger_{\ell+1,w} + e^{+i2\pi\alpha\ell} c^\dagger_{\ell-1,w}) c_{\ell,w}}_{\text{NN inter-cell transfer (accumulates a phase!)}}$$

Transfer Matrices

- The transfer matrix T_ℓ "transfers" the state in cell ℓ the state in cell $\ell+1$
- Define the state of cell ℓ to be $|\ell\rangle = |\ell_1, \ell_2, \ldots, \ell_W\rangle$, where $|\ell_w\rangle \equiv |\ell, w\rangle$

$$\begin{pmatrix} \ell+1 \\ \ell \end{pmatrix} = T_\ell \begin{pmatrix} \ell \\ \ell-1 \end{pmatrix}$$

- One way to construct transfer matrices is to use the Time Independent Schrödinger Equation, $H|\ell\rangle = E|\ell\rangle$, substituting for H, and rearranging to the desired form
- Two transfer matrices are, for $\bar{H}_\ell = H_\ell - (\text{inter-cell transfer})$

$$T_\ell^{\text{anderson}} = \frac{1}{t}\begin{pmatrix} E\mathbf{1} - \bar{H}_\ell & -t\mathbf{1} \\ t\mathbf{1} & 0 \end{pmatrix}; \qquad T_\ell^{\text{hofstadter}} = \frac{1}{te^{-i2\pi\alpha\ell}}\begin{pmatrix} E\mathbf{1} - \bar{H}_\ell & -te^{+i2\pi\alpha\ell}\mathbf{1} \\ te^{-i2\pi\alpha\ell}\mathbf{1} & 0 \end{pmatrix}$$

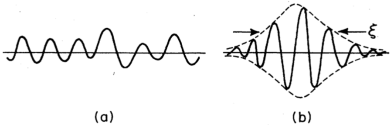
Electronic Localization Length and Conductivity (1 of 3)

- Anderson Localization of electrons in the presence of disorder, assuming the wave function is centered at $|\ell_0, w\rangle$, where $C_{\ell,w} = \langle \ell, w|\ell, w\rangle$, and ξ_w is the "localization length"

$$C_{\ell,w} \approx C_{\ell_0,w} \exp(-|\ell - \ell_0|/\xi_w)$$

- In Ergodic Theory, the Lyapunov Exponent γ, describes the divergence in phase space of two trajectories Z with initial separation $\delta Z(t_0)$ as $|\delta Z(t)| \approx \exp(\gamma|t - t_0|)|\delta Z(t_0)|$
- If we take $\gamma \to -\gamma$ representing convergence of trajectories rather than the divergence, consider the separation from the zero trajectory, and rename $t \to \ell$, $t_0 \to \ell_0$, then:

$$C_{\ell,w} \approx C_{\ell_0,w} \exp(-\gamma_w|\ell - \ell_0|)$$



(a) Extended state (b) Localized state in envelope, $\xi = 1/\gamma$. Lee and Ramakrishnan, Rev. Mod. Phys **57**, 287 (1985).

Electronic Localization Length and Conductivity (2 of 3)

- The best approximation to γ_w occurs at large $|\ell - \ell_0|$, so we consider $\ell = L$ and $\ell_0 = 1$
- Now, Oseledets Theorem[1] states that there exists an asymptotic matrix Γ whose eigenvalues are $\{e^{\pm\gamma_1}, e^{\pm\gamma_2}, \ldots, e^{\pm\gamma_W}\}$, where in our case Γ is defined by:

$$\Gamma = \lim_{L\to\infty} \left(\prod_{\ell=L}^{1} T_\ell^\dagger \prod_{\ell=1}^{L} T_\ell \right)^{1/2L}$$

- By QR decomposition into orthogonal matrix Q and upper triangular matrix R, we have:

$$\prod_{\ell=1}^{L} T_\ell = Q_L \prod_{\ell=1}^{L} R_\ell \quad \Longrightarrow \quad \Gamma = \lim_{L\to\infty} \left(\prod_{\ell=1}^{L} R_\ell \right)^{1/L}$$

- The eigenvalues of R are on the diagonal, so by algebra we have:

$$\gamma_w = \frac{1}{L} \sum_{\ell=0}^{L} \ln |R_\ell^{w,w}|$$

[1] V. Oseledets, Trans. Moscow Math. Soc. **19**, 179 (1968)

Electronic Localization Length and Conductivity (3 of 3)

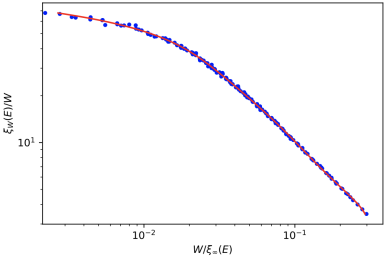
- The most conducting state will dominate the conductivity, so we name a single localization length for a strip of width W as $\xi_W = \max(\xi_w)$.
- The localization length of the infinite system is found by collapsing ξ_W to a function $f(x)$ that fulfills the finite size scaling (FSS) hypothesis, $f(x) = 1/x, x \gg 1$ and $f(x) = \text{const}, x \ll 1$:

$$\frac{\xi_W}{W} = f\left(\frac{W}{\xi_\infty}\right)$$

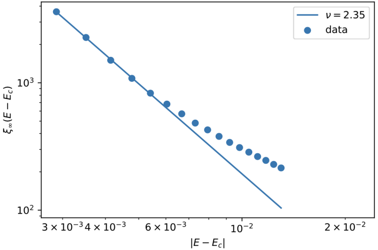
- This localization length diverges around E_c as:

$$\xi_\infty(E) \propto |E - E_c|^{-\nu}$$

Localization length data fits to finite size scaling curve

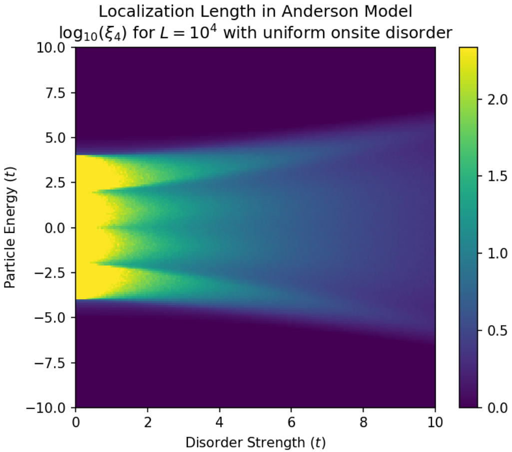


Fit for critical exponent, accuracy of E_c dominates error



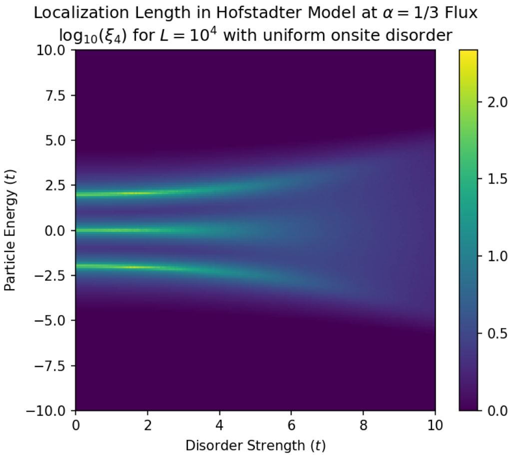
Localization Length in the Anderson Model

- Localization length is attenuated at high disorder
- Symmetric about particle energy $E = 0$
- For disorder strength 1, FSS analysis yields $\nu = 1.00$, however we believe this is not a true divergence and that larger system lengths are needed to yield quantitatively accurate results



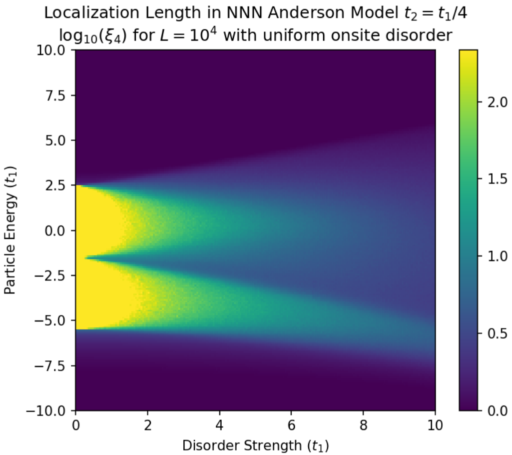
Localization Length in the Hofstadter Model

- Localization length is attenuated at high disorder
- Localization length is suppressed at energies away from three energies where localization length diverges for each disorder strength
- Symmetric about particle energy $E = 0$
- For disorder strength 1, $\nu = 2.47$



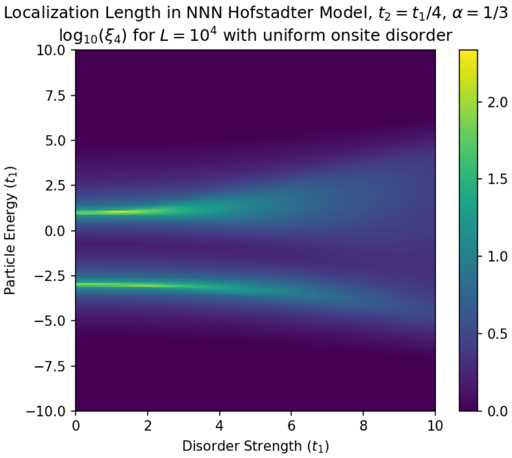
Localization Length in the Next-Nearest Neighbor Anderson Model

- Localization length is attenuated at high disorder
- Asymmetric in particle energy
- Large system lengths are needed to yield quantitatively accurate results



Localization Length in the Next-Nearest Neighbor Hofstadter Model

- Localization length is attenuated at high disorder
- Localization length is suppressed at energies away from two energies where localization length diverges for each disorder strength
- Asymmetric in particle energy



Results

- Qualitative behavior varies significantly depending on the next-nearest neighbor hopping
- Quantitatively, we find $\nu = 2.47 \pm 0.09$ for uniform disorder with strength of the hopping, but this value is non-universal and depends on the disorder strength and the Landau level

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