Mechanics

Forces change the momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a} = m\frac{d\vec{x}}{dt}$$

Forces can come from potentials:

$$\vec{F} = -\vec{\nabla}U$$

Which is equivalent to the work:

$$U = -\int_\ell d\vec{\ell} \cdot \vec{F}$$

Momentum and energy are conserved if they don't enter or leave the system.

Maxwell's Equations

In differential form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \tfrac{1}{c^2}\tfrac{\partial \vec{E}}{\partial t}$$

In integral form:

$$\int_A d\vec{A} \cdot \vec{E} = \int_V \tfrac{\rho}{\epsilon_0} dV$$

$$\oint_\ell d\vec{\ell} \cdot \vec{E} = -\frac{\partial}{\partial t}\int_A d\vec{A} \cdot \vec{B} = -\tfrac{\partial \Phi_B}{\partial t}$$

$$\int_A d\vec{A} \cdot \vec{B} = 0$$

$$\oint_\ell d\vec{\ell} \cdot \vec{B} = \int_A d\vec{A} \cdot \left(\mu_0 \vec{J} + \tfrac{1}{c^2}\tfrac{\partial \vec{E}}{\partial t}\right) = \mu_0 I_{\text{encl}} + \tfrac{1}{c^2}\tfrac{\partial \Phi_E}{\partial t}$$

Vector Algebra

The dot product, or overlap, is defined:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z = \|\vec{u}\|\,\|\vec{v}\|\cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

The cross product, or normal volume:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

where $|\vdots\vdots\vdots|$ indicates the determinant.

Vector Calculus

The gradient of a scalar function f:

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

The divergence of a vector function $\vec{f}$:

$$\vec{\nabla} \cdot \vec{f}(x,y,z) = \frac{\partial \vec{f}\cdot\hat{x}}{\partial x} + \frac{\partial \vec{f}\cdot\hat{y}}{\partial y} + \frac{\partial \vec{f}\cdot\hat{z}}{\partial z}$$

The curl of a vector function $\vec{f}$:

$$\vec{\nabla} \times \vec{f}(x,y,z) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{f}\cdot\hat{x} & \vec{f}\cdot\hat{y} & \vec{f}\cdot\hat{z} \end{vmatrix}$$

Lorentz Force Law

For a point charge:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

For part of a neutrally charged wire:

$$d\vec{F} = I\,d\vec{\ell} \times \vec{B}$$

Note that magnetic fields do no work!

Cyclotron Motion

If $\vec{B} \perp \vec{v}$ then a point charge will gyrate with period $T = 2\pi R/\|\vec{v}\|$ and radius:

$$R = \frac{m\,\|\vec{v}\|}{|q|\,\|\vec{B}\|}$$

If $\vec{B}\cdot\vec{v} = \|\vec{B}\|\,\|\vec{v}\|\cos\theta$, then the radius is $R \mapsto R/\sin(\theta)$ and motion is helical.

Electromagnetic Flux

The electric flux through an area A:

$$\Phi_E = \int_A d\vec{A} \cdot \vec{E}$$

The magnetic flux through an area A:

$$\Phi_B = \int_A d\vec{A} \cdot \vec{B}$$

Biot-Savart Law

The $\vec{B}$-field of a moving charge is:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{q\,\vec{v} \times \vec{r}}{\|\vec{r}\|^3}$$

Or for a differential current element:

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\ell} \times \vec{r}}{\|\vec{r}\|^3}$$

Ampere's Loop Law

Ampere's Law is useful for constant currents with circular symmetry:

$$\oint_\ell d\vec{\ell} \cdot \vec{B} = \mu_0 I_{\text{encl}}$$

Some results found via Ampere's Law:

Current Geometry	Magnetic Field
Long straight wire	$\vec{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\theta}$
Above circular loop	$\vec{B} = \frac{\mu_0 I a^2\,\hat{z}}{2(z^2+a^2)^{3/2}}$
Inside conductor	$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}\,\hat{\theta}$
Center of solenoid	$\vec{B} = \mu_0 N I\,\hat{z}$

Faraday Law & Inductance

There is a voltage (emf) associated with changing magnetic fluxes:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

Let the self-inductance of a current be:

$$L = \Phi_B / I$$

Energy can be stored in magnetic fields:

$$U_L = \tfrac{1}{2}LI^2$$

Circuit Elements

The voltages of circuit components are:

$$\mathcal{E}_{\text{battery}} = \mathcal{E}_0$$

$$\mathcal{E}_{\text{resistor}} = IR$$

$$\mathcal{E}_{\text{capacitor}} = Q/C$$

$$\mathcal{E}_{\text{inductor}} = -L\,dI/dt$$

if a capacitor is in a wire, $I = dQ/dt$.

Kirchhoff's Laws

Voltage is single-valued, so the sum of voltages around any loop is zero:

$$\sum_{\circlearrowleft} \mathcal{E} = 0$$

Charge is conserved (no accumulation), so at any junction currents sum to zero:

$$\sum_{\perp} I = 0$$

Kirchoff's Laws form a system of linear ODEs/equations which can be solved.

Characteristic Equations

A powerful method to solve linear homogeneous differential equations is the method of characteristic equations. Consider the ODE:

$$a\frac{d^2q(t)}{dt^2} + b\frac{dq(t)}{dt} + c\,q(t) = 0$$

The general solution to this is:

$$q(t) = q_+ e^{\omega_+ t} + q_- e^{\omega_- t}$$

where:

$$\omega_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and $q_\pm$ are fixed by boundary values.

Alternating Current

AC circuits are periodically driven:

$$\mathcal{E}_{\text{source}} = \mathcal{E}_0\cos(\omega t)$$

$$I_{\text{source}} = I_0\cos(\omega t)$$

Ohm's Law with complex impedances:

$$\mathcal{E} = IZ$$

Where the impedances are given by:

$$Z_{\text{resistor}} = R$$

$$Z_{\text{capacitor}} = 1/i\omega C$$

$$Z_{\text{inductor}} = i\omega L$$

Series and parallel impedances add as:

$$Z_{\text{series}} = Z_1 + Z_2 + \cdots + Z_n$$

$$Z_{\text{parallel}} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}}$$

If we only care about the amplitudes then we can consider the magnitude:

$$|Z| = \sqrt{\text{Re}(Z)^2 + \text{Im}(Z)^2}$$

Complex Numbers

Let $i = \sqrt{-1}$, and $z = x + iy$ for real numbers x and y, then z is referred to as a complex number. Pictorially, z makes an angle $\arg(z) = \tan^{-1}(y/x)$ with the x axis and is a radius $|z| = \sqrt{x^2+y^2}$ from the origin. We write that $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

The exponential of a complex number:

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

so if $x = -\lambda t$ and $y = \omega t$, then the real part x corresponds to amplitude while the imaginary part y corresponds to oscillatory behavior.

To make a denominator real, multiply by 1 in terms of the complex conjugate:

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

The phase angle of a complex function:

$$\theta(f(t)) = \tan^{-1}\left(\frac{\mathrm{Im}(f(t))}{\mathrm{Re}(f(t))}\right)$$

Electromagnetic Waves

From Maxwell's equations, one can derive that in free space for $1/c^2 = \epsilon_0\mu_0$:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

which corresponds to the propagation of electromagnetic waves at velocity c. A similar relation holds for the magnetic field when scaled as $|\vec{B}| = |\vec{E}|/c$. For electromagnetic waves we have $\hat{E} \perp \hat{B}$, and $\hat{S} = \hat{E} \times \hat{B}$ is the direction of propagation.

The Poynting Vector

Light has momentum and so there is a radiation pressure and force:

$$\vec{F} = \vec{S}\,A/c$$

where $\vec{S}$ is the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{B}/\mu_0$$

Classic Optics

Reflection direction is symmetric with respect to the surface normal.

Snell's law relates the incident light's angle with respect to the surface normal to the refracted light's angle:

$$n_1\sin(\theta_1) = n_2\sin(\theta_2)$$

Total internal reflection occurs when $\theta > \theta_{\text{crit}}$ where:

$$\theta_{\text{crit}} = \sin^{-1}\left(\frac{n_{\text{exterior}}}{n_{\text{interior}}}\right)$$

Wave Interference

Waves of the same wavelength can interfere constructively if $\phi = 2m\pi$ or destructively if $\phi = (2m+1)\pi$ for phase shifts ϕ and integers m. The phase shift accumulated over a length ℓ in a material with index of refraction n is:

$$\phi = 2\pi\ell/\lambda_n$$

where $\lambda_n = \lambda/n$ is the material-specific wavelength and $n = c/v$.

Geometric Optics

Focal lengths are positive or negative:

Optical Element	Focal Length
Convex lens	$f > 0$
Concave lens	$f < 0$
Convex mirror	$f > 0$
Concave mirror	$f < 0$

although no light goes through a mirror.

For multiple thin lenses close together:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + \cdots + \frac{1}{f_n}$$

The image distance is given by:

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f} \iff d' = \left(\frac{1}{f} - \frac{1}{d}\right)^{-1}$$

The image size is given by:

$$s' = -\frac{d'}{d}s$$

Special Relativity

Let there be two frames f and f' oriented in the same direction with origins given by $\vec{o} = \vec{o}' + ut\hat{x}$ for some speed u. An action occurs in the f frame and is observed in the f' frame. If the action in the f frame is described by the 4-vector $\vec{x} = (ct, x, y, z)^\top$, then the observed action in the f' frame is:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where $\beta = u/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

Different behaviors in different frames:

- Time dilation is $t \mapsto t'$
- Length contraction is $x \mapsto x'$
- Velocity distortion is $x/t \mapsto x'/t'$

Relativistic energy is:

$$E = \sqrt{m^2c^4 + p^2c^2}$$

which is $E = mc^2$ if $|p| \ll |mc|$, or $E = pc$ if $|mc| \ll |p|$, where momentum is:

$$\vec{p} = \gamma m\vec{v}$$

INTENTIONALLY LEFT BLANK