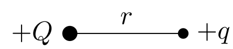
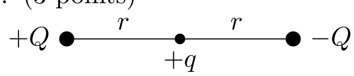
Problem 1. (20 Points)

I. (6 Points)

(a) A positive charge $+q$ is a distance r from a point charge $+Q$. What is the potential energy of $+q$? (3 points)



(b) If a second charge, $-Q$, is added a distance r on the other side of the charge $+q$, the force of $+q$ increases. What is the potential energy of the point charge $+q$? (3 points)

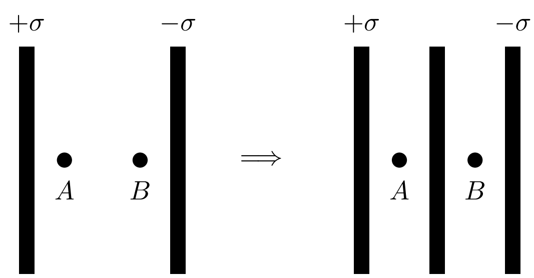


(a) $U = -\int F \cdot d\bar{r} = -\int_\infty^r E\, d\bar{r} = -\frac{qQ}{4\pi\varepsilon_0}\int_\infty^r \frac{1}{\bar{r}^2} d\bar{r} = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{r}$

(b) Energies add, so: $U=0$ (but $F \neq 0$)

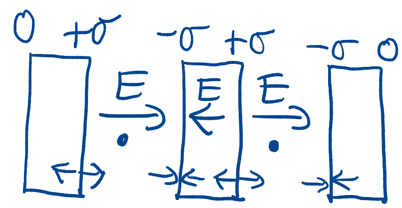
II. (5 points)

Points A and B lie in between two large conducting plates. If a slab of conducting material is inserted as shown below, what happens to the potential difference, $|V_A - V_B|$, between A and B? (5 points)



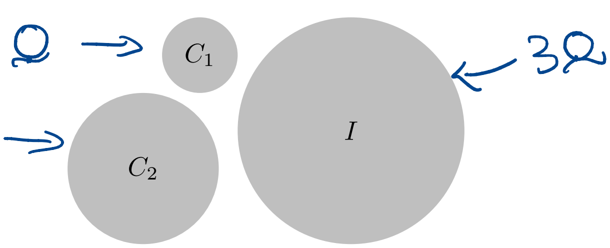
$\Delta V = \left| \int_{r_A}^{r_B} \vec{E} \cdot d\vec{r} \right|$

It will be reduced. Graphically:



III. (9 points)

There are three balls as shown. There are two conductors with radii a and $2a$ and charges $-Q$ and $4Q$ respectively, and one insulator of radius $3a$ and surface charge $3Q$. If we let them touch each other and then separate them simultaneously, what are the charges on C_1 (3 points), C_2 (3 points), and I (3 points)?



The electric potential is the same everywhere on the surface of a conductor, so:

$V = \frac{1}{4\pi\varepsilon_0}\frac{3Q}{3R} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} \Rightarrow Q_{c_1} = Q,\ Q_{c_2} = 2Q$

Problem 2. (20 Points)

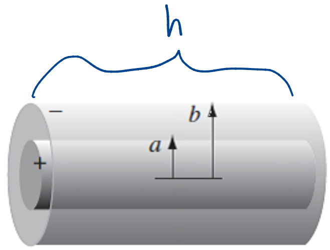
A long coaxial cable carries a uniform volume charge density ρ_0 on the inner cylinder of radius a, and a uniform surface charge density σ on the outer cylinder of radius b. The surface charge is negative, and is of just the right magnitude so that the cable as a whole is electrically neutral.

(a) Find the electric field for $r < a$ (5 points)

(b) Find the electric field for $a < r < b$ (5 points)

(c) Find the electric field for $b < r$ (5 points)

(d) Plot $|\boldsymbol{E}|$ as a function of r (5 points)



We use gauss' law, noting $E \cdot dA = E\,dA$, so $E = \frac{q_{enc}}{\varepsilon_0 A}\hat{r}$.

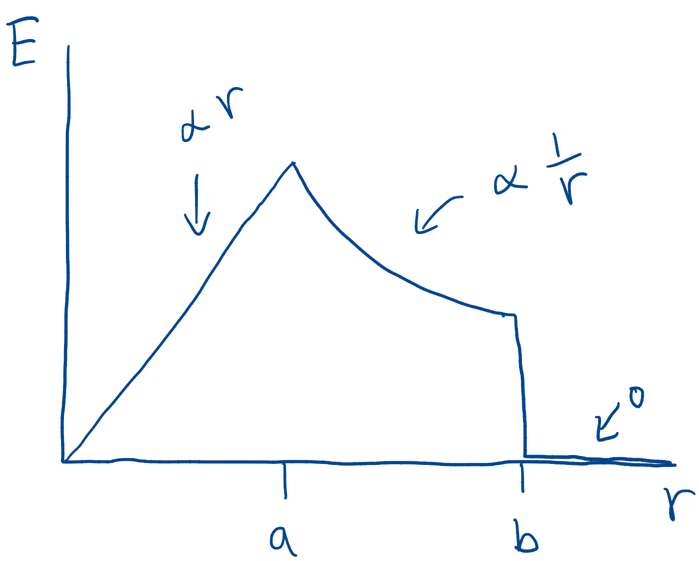
The charge enclosed in the inner cylinder is:

$$q_{cylinder}(r) = \int_{charges} dq = \int_0^a \rho_0\, 2\pi \bar{r} h\, d\bar{r} = \rho \pi r^2 h \overset{r=a}{=} \rho \pi a^2 h$$

(a-c) The electric field is:

$$E(r) = \frac{1}{2\pi r h} \begin{cases} \rho \pi r^2 h & 0 < r < a \\ \rho \pi a^2 h & a < r < b \\ 0 & b < r < \infty \end{cases}$$

(d)



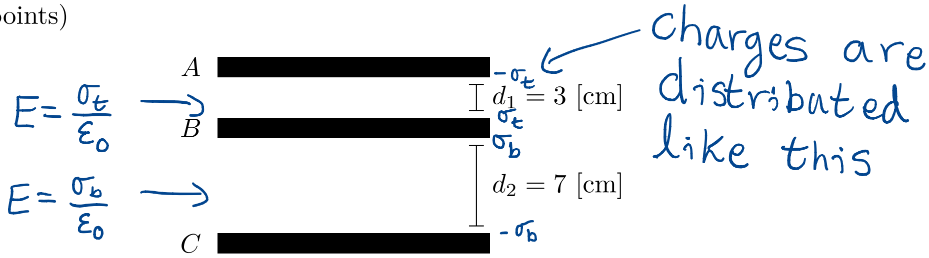
Problem 3. (25 Points)

Three large conducting plates A, B, and C are placed in parallel. Plates A and C are connected using a conducting wire. The inner plate, B, is isolated and carries a total surface charge of $\sigma = 1$ [μC/cm^2]. The charge on plate B will divide itself into a surface charges on the top surface and on the bottom surface: call these charges σ_t and σ_b.

(a) What is the potential difference between A and B? B and C? (express using σ_t and σ_b) (10 points)

(b) What is the relationship between these two potential differences? (5 points)

(c) Find σ_t and σ_b. (10 points)



(a) We note for E constant, $|V(r_1) - V(r_2)| = |E||r_1 - r_2|$, so:

$$|V(B) - V(A)| = \frac{3\sigma_t}{\varepsilon_0}$$

$$|V(B) - V(C)| = \frac{7\sigma_b}{\varepsilon_0}$$

(b) Since A and C are connected with a wire, they are at the same potential, so:

$$V(B) - V(A) = -[V(B) - V(C)]$$

and

$$\frac{3\sigma_t}{\varepsilon_0} = -\frac{7\sigma_b}{\varepsilon_0}$$

(c) Thus, solving the linear equations:

$$3\sigma_t = -7\sigma_b \quad \text{and} \quad \sigma_t + \sigma_b = 1,$$

$$\sigma_t = \frac{7}{4} \left[\frac{\mu C}{cm^2}\right]$$

$$\sigma_b = -\frac{3}{4} \left[\frac{\mu C}{cm^2}\right]$$

Problem 4. (35 Points)

A conducting sphere, B, has radius R_1 and carries total charge $+Q$. Inside of B is a spherical cavity of radius R_0, at the center of which is a point charge, A, of magnitude $+Q$. An insulating shell, C, surrounds the sphere and the point charge. C has inner radius R_2 and outer radius R_3 and carries a charge distribution $\rho(r) = -Q/r^3$.

(a) What charges are on the inner (2 points) and outer (2 points) surfaces of the conducting sphere B? Is there any net charge in the bulk of B? (1 point)

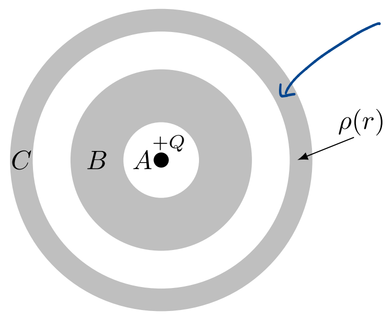
(b) Find the electric field at all points in space: $0 < r < R_0$, $R_0 < r < R_1$, $R_1 < r < R_2$, $R_2 < r < R_3$ and $R_3 < r$. (2 points each)

(c) Plot the electric field as a function of r. (5 points)

(d) Find the electric potential V at all points in space: $0 < r < R_0$, $R_0 < r < R_1$, $R_1 < r < R_2$, $R_2 < r < R_3$ and $R_3 < r$. (V is 0 at infinity). (2 points each)

(e) Plot the electric potential as a function of r. (5 points)

$$q_{shell} = \int_{charge} dq = \int_{R_2}^{r} -4\pi \bar{r}^2 \frac{Q}{\bar{r}^3}\, d\bar{r}$$



$$= -4\pi Q \ln(r/R_2)$$

$$\overset{r=R_3}{=} -4\pi Q \ln(R_3/R_2)$$

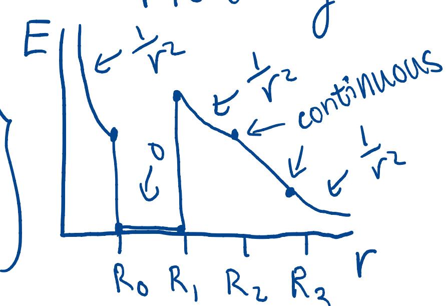
(a) Since there is no total charge in a conductor,

$q_{in} = -Q$, $q_{out} = 2Q$, and $q_{bulk} = 0$

(b) We note that $E \cdot dA = E\,dA$, so $E = \frac{q_{enc}}{\varepsilon_0 A} = \frac{q_{enc}}{4\pi\varepsilon_0 r^2}$, now as a function of r:

$$q_{enc} = \begin{cases} Q & 0<r<R_0 \\ 0 & R_0<r<R_1 \\ 2Q & R_1<r<R_2 \\ 2Q - 4\pi Q \ln(r/R_2) & R_2<r<R_3 \\ 2Q - 4\pi Q \ln(R_3/R_2) & R_3<r<\infty \end{cases}$$

(c) Plotting



(d) We have $V = \int_r^{\infty} E \cdot d\bar{r}$, so upon integration:

$$V = \begin{cases} V(R_0) + Q(1/r - 1/R_0)/4\pi\varepsilon_0 & 0<r<R_0 \\ V(R_1) + 0 & R_0<r<R_1 \\ V(R_2) + Q(1/r - 1/R_2)/2\pi\varepsilon_0 & R_1<r<R_2 \\ V(R_3) + Q(R_3 - r)(1 - 2\pi \ln(r/R_2))/2\pi\varepsilon_0 r R_3 & R_2<r<R_3 \\ Q[1 - 2\pi \ln(R_3/R_2)]/2\pi r \varepsilon_0 & R_3<r<\infty \end{cases}$$

(e) Plotting: