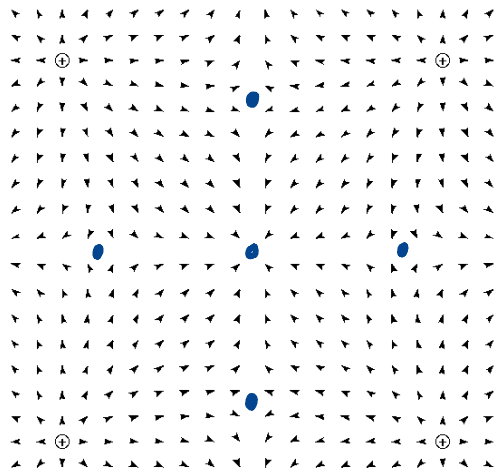
Problem 1. (15 Points)

I. (5 points)

Suppose that four stationary point charges $+Q$ are placed at the corners of a square of side length a as shown below. If a free point charge $+q$ is placed at rest somewhere in the square, at how many distinct points can it stay at rest?

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
(f) 5

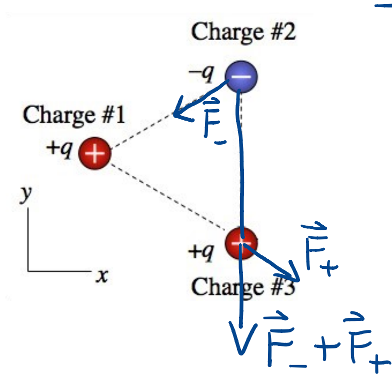


Observe that there are five points where $\sum_{i=1}^{4} \vec{E}_i = 0$. One is a stable equilibrium, the rest are unstable.

II. (5 points)

Three point charges lie at the vertices of an equilateral triangle as shown below. Charges 2 and 3 make up an electric dipole. The net electric force that charge 1 exerts on the dipole is in the:

(a) $+\hat{\boldsymbol{x}}$ direction
(b) $+\hat{\boldsymbol{y}}$ direction
(c) none of the above
(d) $-\hat{\boldsymbol{x}}$ direction
(e) $-\hat{\boldsymbol{y}}$ direction



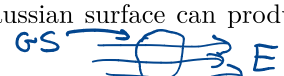
The net force on the dipole for this geometry points down.

III. (5 points)

True/False questions.

(a) Only charge enclosed within a Gaussian surface can produce an electric field at points on the Gaussian surface. (3 points)

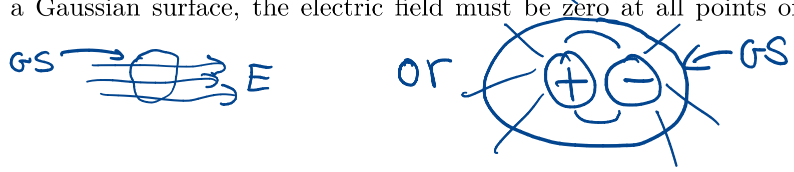
False



or

(b) If there is no net charge inside of a Gaussian surface, the electric field must be zero at all points on the Gaussian surface. (2 points)

False



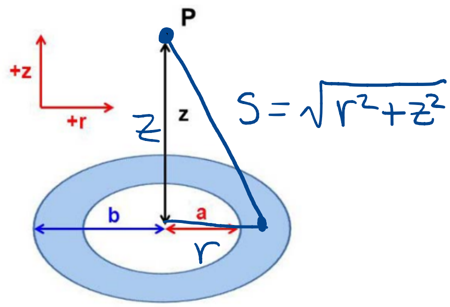
Problem 2. (30 Points)

An annulus has inner radius a and outer radius b and carries a uniform charge density of σ.

(a) Calculate the electric potential V at point P, a distance z above the origin on the positive z-axis. (15 points)

(b) Using $\boldsymbol{E} = -\nabla V$, calculate the electric field at point P from the electric potential V. (10 points)

(c) What is the electric field at P for a uniformly charged disk of radius R? (5 points)



(a) Use the relation:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\text{charge}} \frac{dq}{S} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{\sigma\, 2\pi r\, dr}{\sqrt{r^2+z^2}}$$

with $u = r^2+z^2$, $dr = du/2r$, so

$$V = \frac{\sigma}{4\varepsilon_0}\int_{a^2+z^2}^{b^2+z^2} \frac{du}{\sqrt{u}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)$$

(b) Use the relation:

$\vec{E} = -\vec{\nabla} V$, where here $\vec{\nabla} f = \frac{\partial f}{\partial z}\hat{z}$, so

$$\vec{E} = \frac{\partial}{\partial z}\left[\frac{\sigma}{2\varepsilon_0}\left(\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right)\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{z^2+a^2}} - \frac{z}{\sqrt{z^2+b^2}}\right]\hat{z}$$

(c) This is the case of $a=0$, $b=R$, which is:

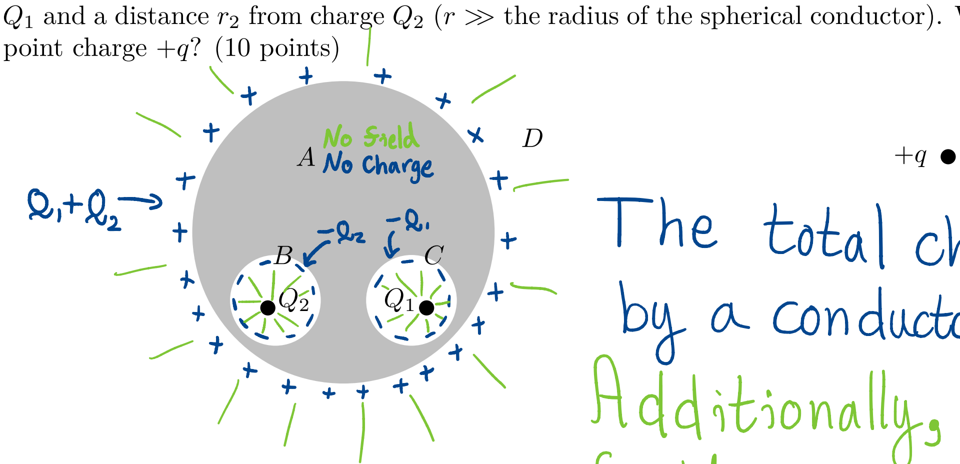
$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{z}$$

Problem 3. **(20 Points)**

Two off-centered cavities are located inside a spherical conductor. Two off-centered point charges $+Q_1$ and $+Q_2$ are located inside these cavities as shown.

(a) Qualitatively, draw the surface charge distributions and electric field lines at A, B, C and D. (10 points)

(b) A point charge $+q$ is placed outside the conductor a distance r from the center of the conductor, a distance r_1 from charge Q_1 and a distance r_2 from charge Q_2 ($r \gg$ the radius of the spherical conductor). What is the total force acting on the point charge $+q$? (10 points)



The total charge enclosed by a conductor is 0.

Additionally, the electric field is perpendicular to the surface of a conductor.

(b) Since r is large, the whole charge distribution looks like a monopole of magnitude Q_1+Q_2 a distance r away:

$$\vec{F} = q\vec{E} = q \cdot \frac{1}{4\pi\varepsilon_0} \frac{Q_1+Q_2}{r^2} \hat{r}$$

Problem 4. (35 Points)

A hollow insulating spherical shell of inner radius R_0 and outer radius R_1 carries a charge density of $\rho(r) = \rho_0(r/R_1)^3$. A positive charge $+Q$ is placed in the center of the hollow spherical shell and a grounded conducting shell with inner radius R_2 and outer radius R_3 surrounds the hollow sphere.

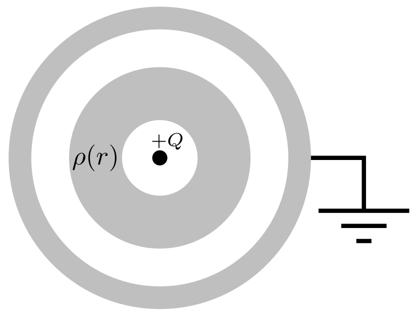
(a) What is the total charge on the insulating spherical shell? (5 points)

(b) What charges are on the inner and outer surfaces of the conducting shell? (5 points)

(c) Find the electric field at all points in space. (15 points)

(d) Plot the electric field as a function of r. (5 points)

(e) How would change if the conducting shell was not grounded (and was not given any charge)? (5 points)



(a) We find the charge enclosed as dependent on r:

$$q_{shell}(r) = \int_{charges} dq = \int_{R_0}^{r} \rho_0 \frac{\bar{r}^3}{R_1^3} 4\pi \bar{r}^2 d\bar{r} = \frac{2\pi\rho_0}{3} \frac{r^6 - R_0^6}{R_1^3} \overset{r=R_1}{=} \frac{2\pi\rho_0}{3} \frac{R_1^6 - R_0^6}{R_1^3}$$

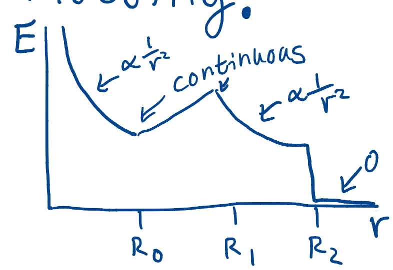
(b) There is no total charge enclosed in a conductor, and a ground has $V=0$, hence:

$$q_{in} = -\left(Q + q_{shell}(R_1)\right) \quad \text{and} \quad q_{out} = 0$$

(c) We note $E \cdot dA = E dA$, so $E = q_{enc}/\varepsilon_0 A = q_{enc}/4\pi\varepsilon_0 r^2$, and

$$E = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \begin{cases} Q & 0 < r < R_0 \\ Q + \frac{2\pi\rho_0}{3} \frac{r^6 - R_0^6}{R_1^3} & R_0 < r < R_1 \\ Q + \frac{2\pi\rho_0}{3} \frac{R_1^6 - R_0^6}{R_1^3} & R_1 < r < R_2 \\ 0 & R_2 < r < \infty \end{cases}$$

(d) Plotting:



(e) If the conductor was neutral, then $q_{out} = -q_{in}$, and

So for $R_3 < r < \infty$, $q_{enc} = Q + \frac{2\pi\rho_0}{3} \frac{R_1^6 - R_0^6}{R_1^3}$