Problem 1. Charged Cylinder

Consider a long insulating cylinder of radius R and charge density $\rho(r) = \rho_0 r$, surrounded by a neutrally charged conducting shell of inner radius $2R$ and outer radius $3R$.

- What is the enclosed charge as a function of radius?
- What is the electric field as a function of radius?
- What is the electric potential as a function of radius?

Let us calculate the charge enclosed for $0 \le r \le R$:

$$q(r) = \int_0^r dq = \int_0^r \rho dV = \int_0^r \rho_0 \bar{r}\ 2\pi \bar{r} h d\bar{r} = 2\pi\rho_0 h \int_0^r \bar{r}^2 d\bar{r} = \frac{2\pi\rho_0 h}{3} r^3$$

For $R < r < 2R$:

$$q(r) = \int_0^r dq = \int_0^R dq + \int_R^r dq = \frac{2\pi\rho_0 h}{3} R^3 + \int_R^r dq = \frac{2\pi\rho_0 h}{3} R^3$$

For $2R < r < 3R$:

$$q(r) = \int_0^r dq = \int_0^R dq + \int_R^{2R} dq + \int_{2R}^r = \frac{2\pi\rho_0 h}{3} R^3 + 0 + \int_{2R}^r dq = \frac{2\pi\rho_0 h}{3} R^3 - \frac{2\pi\rho_0 h}{3} R^3 = 0$$

For $3R < r < \infty$:

$$q(r) = \int_0^r dq = \int_0^R dq + \int_R^{2R} dq + \int_{2R}^{3R} dq + \int_{3R}^r dq = (0) + \int_{3R}^r dq = (0) + \frac{2\pi\rho_0 h}{3} R^3 = \frac{2\pi\rho_0 h}{3} R^3$$

Electric fields then follow naturally by assuming the electric field is parallel to the surface normal vector ($\boldsymbol{E} \cdot d\boldsymbol{A} = EdA$), where we can insert $q(r)$ from above:

$$\boldsymbol{E}(r) = \frac{q(r)}{\epsilon_0 A(r)} \hat{\boldsymbol{r}} = \frac{q(r)}{\epsilon_0 2\pi r h} \hat{\boldsymbol{r}}$$

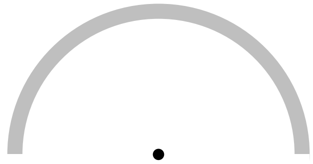
The potential is at $r > 3R$:

$$V(r) = \int_r^\infty \boldsymbol{E} \cdot d\bar{\boldsymbol{r}} = \int_r^\infty E d\bar{r} = \int_r^\infty \frac{2\pi\rho_0 h}{3\ \epsilon_0 2\pi \bar{r} h} R^3 d\bar{r} = \frac{\rho_0}{3\epsilon_0} R^3 \int_r^\infty \frac{d\bar{r}}{\bar{r}} = \frac{\rho_0}{3\epsilon_0} R^3 \ln(\bar{r}) \Big|_{\bar{r}=r}^{\bar{r}=\infty} = \infty$$

This broke down since we implicitly assumed that $h = \infty$ when we assumed that $\boldsymbol{E} \cdot d\boldsymbol{A} = EdA$. This means that we accidentally considered an infinite distribution of charge. Solving for finite h is rather harder.

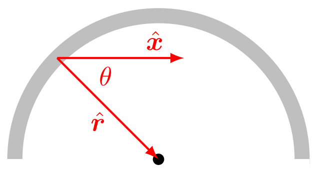
Problem 2. Half of a Ring

Consider a half ring of radius a centered on the origin with charge $-q$ from polar angle 0 to $\pi/2$ and charge $+q$ from polar angle $\pi/2$ to π. What is the electric field at the origin?



Here we begin by drawing the field lines **on scratch paper**, and note that from symmetry, that at the origin, the electric field points exclusively in the $+\hat{\boldsymbol{x}}$ direction (to the right). This means that $\boldsymbol{E} = \boldsymbol{E}_x \hat{\boldsymbol{x}} = (\boldsymbol{E} \cdot \boldsymbol{x})\hat{\boldsymbol{x}} = E\cos(\theta)\hat{\boldsymbol{x}}$.[1]

Next, we parametrize the semicircle with the parameter θ. Here, θ **is not** the angle in polar coordinates, but rather the angle between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{x}}$:



Now, we calculate the linear charge density, $\lambda = Q/L$:

$$\lambda(\theta) = \begin{cases} 0 \leq \theta < \pi/2 & +2q/\pi a \\ \pi/2 < \theta \leq \pi & -2q/\pi a \end{cases}$$

The electric field in the x direction is then, with $dq = a d\theta\ \lambda(\theta)$:

$$E_x = \int_0^\pi \frac{k\ a d\theta\ \lambda(\theta)}{a^2}\cos(\theta) = \int_0^\pi \frac{k\lambda(\theta)}{a}\cos(\theta)d\theta = \int_0^{\pi/2} \frac{2qk}{\pi a^2}\cos(\theta)d\theta - \int_{\pi/2}^{\pi} \frac{2qk}{\pi a^2}\cos(\theta)d\theta = \frac{2qk}{\pi a^2} - -\frac{2qk}{\pi a^2} = \frac{4kq}{\pi a^2}$$

So, the electric field is:

$$\boldsymbol{E}(0,0) = \frac{4kq}{\pi a^2}\hat{\boldsymbol{x}}$$

[1] This all works out nicely because cosine is an even function

Problem 3. Point Charge

Consider a point charge Q inside an insulating shell of inner radius R and outer radius $2R$ with total charge q. What is the potential everywhere if the potential at infinity is 0?

Begin by finding the enclosed charge as a function of radius. For $0 < r < R$:

$$q(r) = \int_0^r dq = Q$$

For $R \leq r \leq 2R$:

$$q(r) = \int_0^r dq = \int_0^R dq + \int_R^r dq = Q + \int_R^r \rho dV = Q + \int_R^r \frac{q}{\frac{4}{3}\pi((2R)^3 - (R)^3)} 4\pi\bar{r}^2 d\bar{r} = Q + \frac{3q}{7R^3}\int_R^r \bar{r}^2 d\bar{r} = Q + \frac{q}{7R^3}(r^3 - R^3)$$

For $2R < r < \infty$:

$$q(r) = \int_0^r dq = \int_0^R dq + \int_R^{2R} dq + \int_{2R}^r dq = Q + q$$

Here we have spherical symmetry, and so the electric field is parallel to the surface normal vector ($\boldsymbol{E} \cdot d\boldsymbol{A} = EdA$). Thus the electric field is in terms of the charges calculated above:

$$\boldsymbol{E}(r) = \frac{q(r)}{\epsilon_0 A(r)}\hat{\boldsymbol{r}} = \frac{q(r)}{\epsilon_0 4\pi r^2}\hat{\boldsymbol{r}}$$

The potential in the region $2R < r < \infty$ is, letting $\boldsymbol{E} \cdot d\boldsymbol{r}$ be:

$$V(r) = \int_r^\infty d\bar{r}\ E = \int_r^\infty d\bar{r}\ \frac{Q+q}{\epsilon_0 4\pi\bar{r}^2} = \frac{Q+q}{4\pi\epsilon_0}\int_r^\infty d\bar{r}\ \frac{1}{\bar{r}^2} = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{r}$$

For $R \leq r \leq 2R$:

$$\begin{aligned} V(r) &= \int_r^\infty d\bar{r}\ E = \int_r^{2R} d\bar{r}\ E + \int_{2R}^\infty d\bar{r}\ E = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \int_r^{2R} d\bar{r}\ E = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \int_r^{2R} d\bar{r}\ \frac{Q + \frac{q}{7R^3}(\bar{r}^3 - R^3)}{4\pi\epsilon_0\bar{r}^2} \\ &= \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \frac{1}{4\pi\epsilon_0}\int_r^{2R} d\bar{r}\left[Q\frac{1}{\bar{r}^2} + \frac{q}{7R^3}\bar{r} - \frac{q}{7}\frac{1}{\bar{r}^2}\right] = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \frac{1}{4\pi\epsilon_0}\left[(Q - q/7)\left(\frac{1}{r} - \frac{1}{2R}\right) + \frac{q}{14R^3}(4R^2 - r^2)\right] \end{aligned}$$

We calculate at $r = R$, to simplify and save space:

$$V(R) = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \frac{1}{4\pi\epsilon_0}\left[(Q - q/7)\left(\frac{2}{2R} - \frac{1}{2R}\right) + \frac{q}{14R^3}(4R^2 - R^2)\right] = \frac{Q+q}{4\pi\epsilon_0}\frac{1}{R} + \frac{1}{4\pi\epsilon_0}\frac{1}{R}\left[\frac{7Q+2q}{14}\right] = \frac{1}{4\pi\epsilon_0}\frac{1}{R}\left[\frac{21Q+16q}{14}\right]$$

For $0 < r < R$:

$$\begin{aligned} V(r) &= \int_r^\infty d\bar{r}\ E = \int_r^R d\bar{r}\ E + \int_R^{2R} d\bar{r}\ E + \int_{2R}^\infty d\bar{r}\ E = \frac{1}{4\pi\epsilon_0}\frac{1}{R}\left[\frac{21Q+16q}{14}\right] + \int_r^R d\bar{r}\ E \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{R}\left[\frac{21Q+16q}{14}\right] + \int_r^R d\bar{r}\ \frac{Q}{4\pi\epsilon_0\bar{r}^2} = \frac{1}{4\pi\epsilon_0}\frac{1}{R}\left[\frac{21Q+16q}{14}\right] + \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{7Q+16q}{14R} + \frac{Q}{r}\right) \end{aligned}$$

Problem 4. Conducting Plates

Consider three conducting plates with charge densities 5σ, 3σ and $-\sigma$. Then place the plates in the following arrangement:

What is the charge on each surface? What is the electric field everywhere?

Here we use the ansatz provided in class, for any n parallel conducting plates, the net charge will distribute evenly between the outer edges of the two bounding plates, while each setter of inner sides separated by vacuum have equal and opposite charges (with an associated minimum of the potential energy). So, the distribution of charges for n plates is:

$\sigma_1\ \sigma_2 \quad -\sigma_2\ \sigma_3 \quad -\sigma_3\ \sigma_4 \quad -\sigma_4\ \sigma_5 \quad -\sigma_5\ \sigma_6 \quad \cdots \quad -\sigma_n\ \sigma_1$

First:

$$2\sigma_1 = \sum \sigma_{\text{given}}$$

Then:

$$\begin{aligned}
\sigma_1 + \sigma_2 = \sigma_{\text{given},1} &\implies \sigma_2 = \sigma_{\text{given},1} - \sigma_1 \\
\sigma_3 - \sigma_2 = \sigma_{\text{given},2} &\implies \sigma_3 = \sigma_2 + \sigma_{\text{given},2} \\
\sigma_4 - \sigma_3 = \sigma_{\text{given},3} &\implies \sigma_4 = \sigma_3 + \sigma_{\text{given},3} \\
\sigma_5 - \sigma_4 = \sigma_{\text{given},4} &\implies \sigma_5 = \sigma_4 + \sigma_{\text{given},4} \\
\sigma_6 - \sigma_5 = \sigma_{\text{given},5} &\implies \sigma_6 = \sigma_5 + \sigma_{\text{given},5} \\
&\qquad \cdots
\end{aligned}$$

Inside every conductor, the electric field is 0. Now, each inner pair of surfaces forms a capacitor, so the field is $2\sigma_i/2\epsilon_0 = \sigma_i/\epsilon_0$. By the vector addition of electric fields, outside the capacitor the field is $2\sigma_1/2\epsilon_0 = \sigma_1/\epsilon_0$. For directions, a coordinate system and charges must be specified.

Our specific case is simpler:

$$\sigma_1 = \frac{1}{2}(5\sigma + 3\sigma - \sigma) = \frac{7}{2}\sigma$$

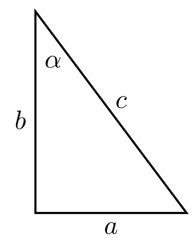
$$\sigma_2 = 5\sigma - \frac{7}{2}\sigma = \frac{3}{2}\sigma$$

$$\sigma_3 = \frac{3}{2}\sigma + 3\sigma = \frac{9}{2}\sigma$$

The electric fields are then as above, substituting for σ_i. If we assume $\sigma > 0$, and let the $+\hat{\boldsymbol{x}}$ direction to be towards the right, then all fields point in the $+\hat{\boldsymbol{x}}$ direction except the field outside and to the left, which points in the $-\hat{\boldsymbol{x}}$ direction.

Problem 5. Short Questions

1. Given the following triangle, what is $\cos(\alpha)$, what is $\sin(\alpha)$?



$$\cos(\alpha) = b/c, \qquad \sin(\alpha) = a/c$$

2. If $V = k(x^2 + y^2 + z^2)$, calculate the electric field.

$$E(x, y, z) = -2k(x, y, z)^T$$

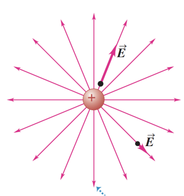
3. What are the SI units of λ, σ, and ρ?

$$[\lambda] = [\mathrm{C/m}], \qquad [\sigma] = [\mathrm{C/m^2}], \qquad [\rho] = [\mathrm{C/m^2}]$$

4. What is the flux from a point charge q through (a) a spherical Gaussian surface of radius r (b) a spherical Gaussian surface of radius $2r$?

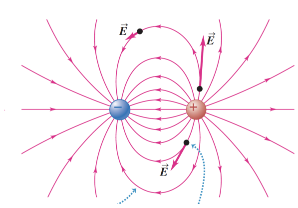
$$q/\epsilon_0, \qquad q/\epsilon_0$$

5. Draw the electric field lines for a point charge? What does this mean in the context of a Gaussian surface that does not enclose the charge?



There is no net flux, yet there is a field inside the Gaussian surface.

6. Draw the electric field lines for a dipole. What does this mean in the context of a Gaussian surface that encloses the dipole?



While there is no net flux, the electric field on the surface is not constant.

7. If the electric field at a distance r from a point charge has magnitude E, what is the magnitude of the electric field at a distance of $10r$? What does this mean for the potential energy of a point charge?

$$E/100$$

8. If the electric field at a distance r from an infinite sheet of conducting charge has magnitude E, what is the magnitude of the electric field at a distance of $10r$? What does this mean for the potential energy of a point charge?

$$E$$

9. Given a square of charges q at $(0, a)$, $(a, 0)$, $(0, -a)$, and $(-a, 0)$, find the electric potential V at $(0, 0)$.

$$V(0, 0) = 4kq/a$$

10. Given a square of charges q at $(0, a)$, $(a, 0)$, $(0, -a)$, and $(-a, 0)$, find the electric field $\boldsymbol{E}$ at $(0, 0)$.

$$E = 0$$

11. What is the electric field E_z on the xy plane given by two charges z located at $(0, 0, 1)^T$, $(0, 0, -1)^T$?

$$E_z = 0$$

12. There is no net charge inside any conductor (True/False)

True

13. There is no net charge inside any insulator (True/False)

False

14. What is the electric field inside a thin conducting shell of radius r with a total charge q?

$$E = 0, \text{ because } V \text{ is constant}$$

15. What is the electric field in between two infinite sheets of surface charge density σ? What is the electric field outside?

$$\text{Inside: } E = 0, \qquad \text{Outside: } E = \sigma/\epsilon_0$$

16. Fluorine, Chlorine, Bromine, and Iodine are gas, gas, liquid, and solid at STP. These elements have the same outer electron configurations of p^5, yet their states of matter are due to electronic behavior how can this be?

The larger the atom, the more electrons it has, so the larger its polarizability and its dipole moment. Sufficiently large dipole moments can convince substances to condense into liquids and solids.