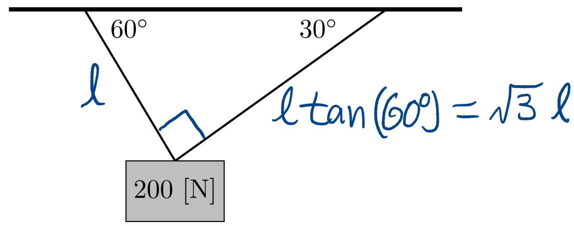
Problem 1. (30 Points)

(a) Two strings made from the same wire and a weight of 200 [N] are set up as shown in the figure. The left string has a mass of 8 [g] and a length of 40 [cm]. If both of them are simultaneously plucked at the center, what is the ratio between the frequencies of the 2nd overtone of the left string to the 1st overtone of the right string? (10 points)

Recall $v = \lambda f$,

and $v = \sqrt{\frac{T}{\mu}}$



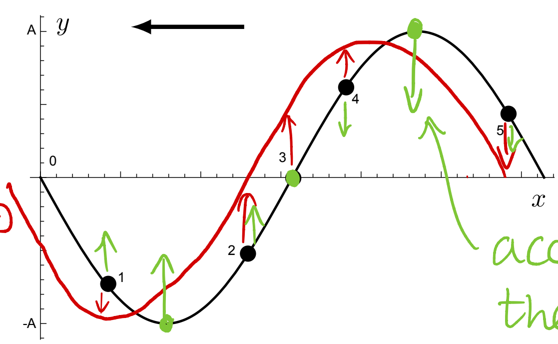
$$\frac{f_L}{f_R} = \frac{v_L \lambda_R}{v_R \lambda_L} = \frac{\sqrt{T_L}\ \lambda_R}{\sqrt{T_R}\ \lambda_L} = \sqrt{\sqrt{3}}\ \frac{\sqrt{3}\, l}{\frac{2}{3} l} = \frac{1}{2}\, 3^{7/4} \approx 3.42$$

$\rightarrow \lambda_L = \frac{2}{3} l$, $\lambda_R = l_R = \sqrt{3}\, l \leftarrow$

The horizontal components of the tensions balance, so:

$$T_L \sin(30°) = T_R \sin 60° \implies \frac{T_L}{T_R} = \frac{\sin(60°)}{\sin(30°)} = \sqrt{3}$$

(b) A transverse wave is traveling in the $-x$ direction. Draw the direction of the velocity and acceleration at points 1, 2, 3, 4 and 5. (6 points)

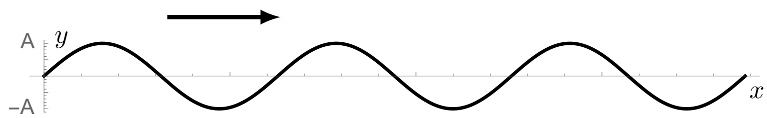


a slightly later time

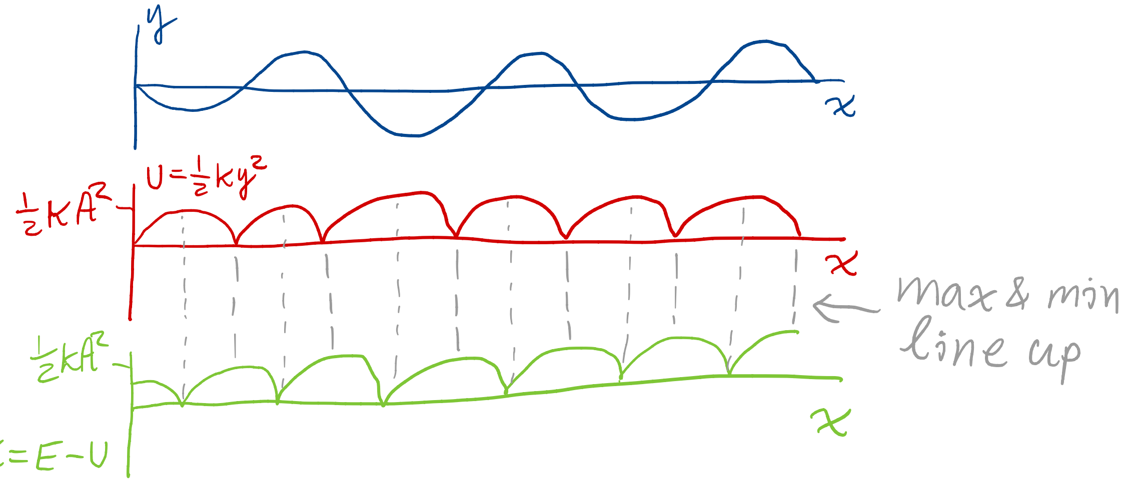
velocity
acceleration

acceleration is in the direction of F

(c) A wave traveling to the right on a string is shown at time $t = 0$. The density of the string is μ, the angular frequency is ω, and the amplitude of displacement is A. Plot the potential energy density and kinetic energy density with respect to the location of the string at $t = \pi/\omega$. (7 points)

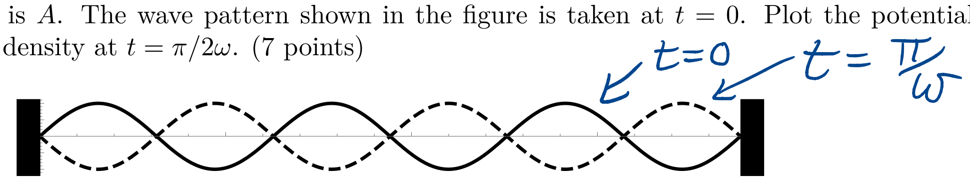


time $t = \frac{\pi}{\omega}$ corresponds to π phaseshift, so:

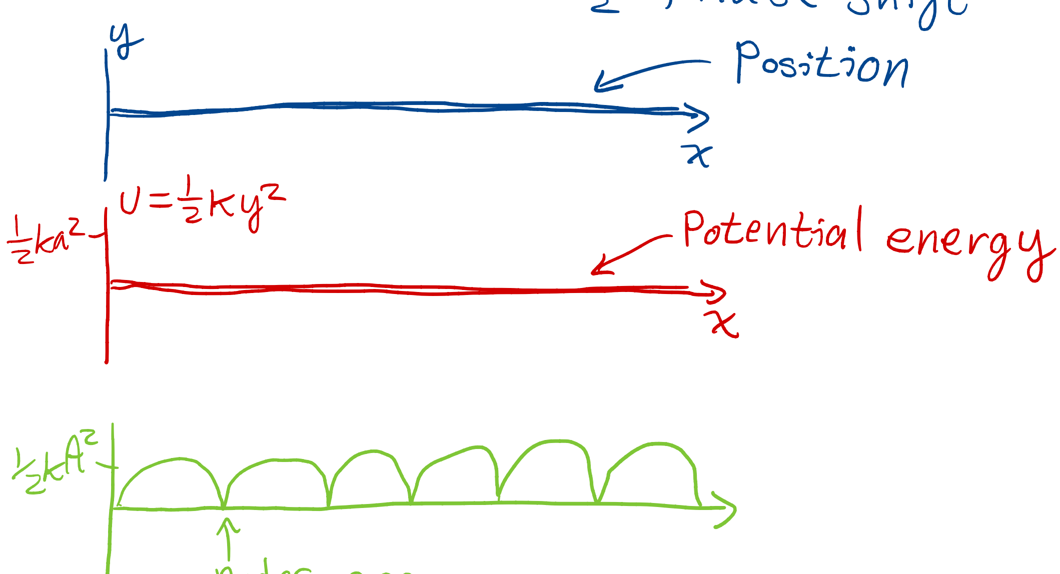


Two ways:
(1) take dy/dt
(2) $E = K + U \Rightarrow K = E - U$

(d) Two ends of a string are clamped on the wall. Then density of the string is μ, the angular frequency is ω, and the amplitude of displacement is A. The wave pattern shown in the figure is taken at $t = 0$. Plot the potential energy density and kinetic energy density at $t = \pi/2\omega$. (7 points)



time $t = \frac{\pi}{2\omega}$ corresponds to $\frac{\pi}{2}$ Phase shift



observation: the energy per unit length is constant for the traveling wave and variable for the standing wave

Problem 2. (20 Points)

A loudspeaker is adjustable from 1 [kHz] to 2 [kHz]. It is placed next to a stopped pipe of length 70.2 [cm]. Both of them remain stationary. The speed of sound in air is 344 [m/s].

(a) If I increase the frequency from 1000 [Hz], at what frequency do I first hear resonance? (5 points)

(b) Plot the displacement normal mode for this frequency. (5 points)

(c) Mark the points where a listener cannot hear anything in this normal mode. (5 points)

(d) What are all the possible resonant frequencies generated by this speaker? (5 points)

(a) Stopped pipes have resonance at standing wavelengths

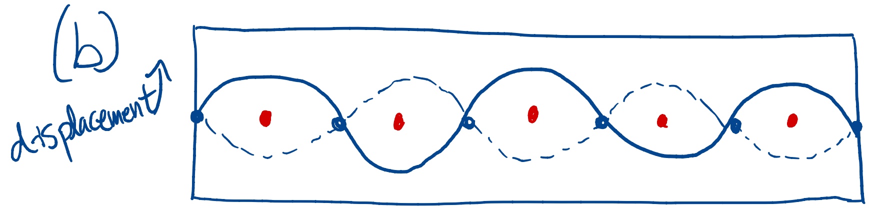
$$\lambda = \frac{2l}{n} \quad \text{for } n = 1, 2, 3, \ldots$$

Additionally, $v = \lambda f$, so the resonant frequencies are:

$$f = \frac{v}{\lambda} = n\frac{v}{2l} = 245n$$

$n = 5 \rightarrow 1225$ [Hz]

(b)

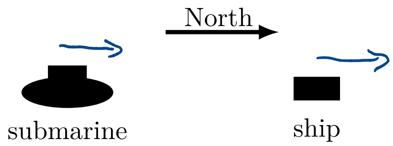


(c) The human ear responds to the applied force of a sound wave, or the pressure variations, so at the pressure nodes/displacement antinodes, one hears nothing.
Marked in red

(d) $n = 5 \rightarrow 1225$ [Hz]
$n = 6 \rightarrow 1470$ [Hz]
$n = 7 \rightarrow 1715$ [Hz]
$n = 8 \rightarrow 1960$ [Hz]

Problem 3. (20 Points)

A submarine floating near to the surface of the ocean is moving north with a speed of 75.2 [km/h] with respect to the ocean floor. It emits a sonar signal (the speed of sound in seawater is 1522 [m/s]) of frequency 989 [Hz]. The ocean has a current moving north at 30.5 [km/h] relative to the ocean floor. The sonar signal is reflected by a ship to the north of the submarine. If the submarine receives a beats frequency of 7 [Hz] and the relative motion between the submarine and the ship brings them apart, what is the velocity of the ship relative to the submarine? (20 points)



We use the doppler effect:

$$f_{emit} = 989\,[Hz]$$

$$f_{hear}^{ship} = \frac{v_{sound}^{N} - v_{ship}}{v_{sound}^{N} - v_{sub}} f_{emit}$$

moving in direction of sound propagation

moving against direction of sound propagation

$$f_{sub}^{hear} = \frac{v_{sound}^{S} + v_{sub}}{v_{sound}^{S} + v_{ship}} f_{hear}^{ship}$$

We recall $f_{Beats} = |f_1 - f_2|$, so:

$$7 = \left| f_{emit} - \frac{v_{sound}^{S} + v_{sub}}{v_{sound}^{S} + v_{ship}} \cdot \frac{v_{sound}^{N} - v_{ship}}{v_{sound}^{N} - v_{sub}} f_{emit} \right|$$

Now:

$$v_{sound}^{S} = 1522 - \frac{30.5}{3600} \cdot 1000 = 1513.5\,[m/s]$$

$$v_{sound}^{N} = 1522 + \frac{30.5}{3600} \cdot 1000 = 1530.5\,[m/s]$$

$$v_{sub} = \frac{75.2 - 30.5}{3600} \cdot 1000 = 11.6\,[m/s]$$

So, solving the equation: $v_{ship} = 17.0\,[m/s]$

Or relative to the sub: $5.4\,[m/s]$

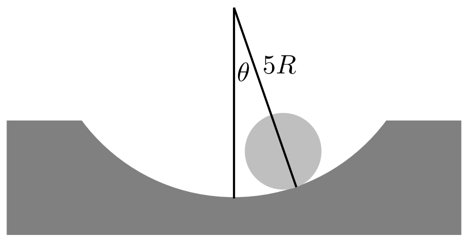
Problem 4. (30 Points)

A solid sphere (radius R) with mass M rolls without slipping in a cylindrical trough (radius $5R$) as shown in the figure.

(a) Express the kinetic energy of the ball in terms of $\dot{\theta}$, R, and M. Use $K = 1/2Mv^2 + 1/2I\omega^2$, where ω is the angular velocity of the ball about its center of mass. Hint: $I_{\text{ball}} = 2/5MR^2$. If there is no slipping, $v = R\omega$. (10 points)

(b) Assuming θ is small, express the potential energy in terms of θ, R, and M. (10 points)

(c) Using the conservation of energy, that is $dE/dt = 0$, show that the motion is simple harmonic motion, and find the period. (10 points)



(a) We are familiar with the expression $\omega = \dot{\theta}$, however the sphere makes four rotations per cycle of θ, so

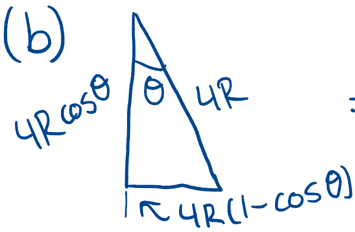
$$\omega = 4\dot{\theta}$$

Thus:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\left(R\,4\dot{\theta}\right)^2 + \frac{1}{2}\frac{2}{5}mR^2\left(4\dot{\theta}\right)^2$$

$$= \frac{112}{10}mR^2\dot{\theta}^2$$

(b)



$$\Rightarrow U = mgh = 4Rmg(1-\cos\theta)$$

$$\approx 2Rmg\theta^2 \leftarrow \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \ldots$$

(c) $$\frac{dE}{dt} = 0 = 2\left[\frac{112mR^2}{10}\dot{\theta}\ddot{\theta} + 2Rmg\theta\dot{\theta}\right]$$

Or,

$$0 = 28R\ddot{\theta} + 5g\theta \Rightarrow \ddot{\theta} = -\frac{5}{28}\frac{g}{R}\theta$$

Which is the simple harmonic oscillator with:

$$\omega = \sqrt{\frac{5}{28}\frac{g}{R}} \Rightarrow T = \frac{1}{f} = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{7}{5}\frac{R}{g}}$$