Problem 1. (25 Points)

A cylindrical bucket, 10 [cm] in diameter, with its top open to the atmosphere, $P = 10^5$ [Pa], is filled to a height of 30.0 [cm] with water. Note that $\rho_{\text{water}} = 1000$ [kg/m^3].

(a) What is the pressure at the bottom of the bucket? (5 points)

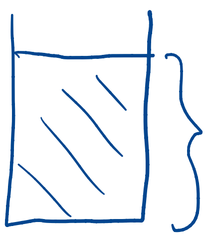
(b) If we place a lead ball of mass 11 [kg] inside the water and then measure the weight of the ball using a scale under the water, what does the scale read? Note $\rho_{\text{lead}} = 11000$ [kg/m^3]. (10 points)

(c) We take out the lead ball and drill a circular hole with an area of 1.25 [cm^2] in the bottom of the bucket. We then add a hose that pours water into the bucket at a rate of 2.5×10^{-4} [m^3/s].

(i) How fast must the water flow through the hole in the bucket to keep the water level constant? (5 points)

(ii) After a long time, what is the height of water in the bucket? (5 points)

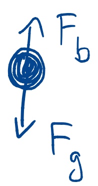
(a)



0.3 m

$$P = P_{atm} + \rho g h$$
$$= 10^5 + 10^3 \cdot 10 \cdot 0.3$$
$$= 1.03 \times 10^5 \text{ [Pa]}$$

(b) Scale measures sum of forces



$$\sum_i F_i = F_g - F_b = mg - \rho_f V g$$
$$= V(\rho_b - \rho_f) g \quad \leftarrow \quad V = \frac{m}{\rho_b}$$
$$= mg \cdot \frac{\rho_b - \rho_f}{\rho_b}$$
$$= \text{Perceived } 10 \text{ [kg]}$$

(c) $\dot{V}_{in} = \dot{V}_{out}$

$$2.5 \times 10^{-4} = 1.25 \times 10^{-4} v$$
$$v = 2 \text{ [m/s]}$$

(d) Equal flow in as out; energy conserved

$$E_{in} = E_{out}$$
$$mgh_{in} = \frac{1}{2} m v_{out}^2 \Rightarrow h_{in} = \frac{v_{out}^2}{2g} = 0.2 \text{ [m]}$$

Problem 2. **(20 Points)**

A spring with spring constant k is displaced a distance x by a force F.

(a) If the spring is cut into three pieces of equal length, how far does one of these pieces stretch under F? (3 points)

(b) What is the spring constant of each piece? (7 points)

(c) If we hang the pieces in parallel and attach a mass m to the end of them, what is the oscillation frequency? (10 points)

(a) $x/3$ (as a whole spring they still experience F)

(b) $K_{piece} \cdot \frac{x}{3} = F$ (Forces balance)

$Kx = F$

$\Longrightarrow K_{piece} = 3K$

(c) $\sum_i F_i = ma$

$-K_{piece}x - K_{piece}x - K_{piece}x + mg = ma$

$\Longrightarrow \ddot{x} = -\frac{9K}{m}x$ (adjusting origin for mg)

$\omega = \sqrt{\frac{9K}{m}}$

$f = \frac{\omega}{2\pi} = \frac{3}{2\pi}\sqrt{\frac{K}{m}}$

Problem 3. **(15 Points)**

An object with mass 1 [kg] is attached to a spring with one end fixed to a wall and moves frictionlessly on the ground. When it is displaced by 0.6 [m] to the right of its equilibrium position, it has a velocity of 2.2 [m/s] to the right and an acceleration of 8.4 [m/s^2] to the left.

(a) What is the spring constant? (5 points)

(b) What is the amplitude? (5 points)

(c) How much further from this point will the object move before it stops and begins to move back to the left? (5 points)

(a) $F = ma$

$-kx = ma$

$\Rightarrow \quad k = -\frac{ma}{x}$

$= -\frac{1 \cdot -8.4}{0.6}$

$= 14 \left[\frac{N}{m}\right]$

(b) Use conservation of energy

$E_0 = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2$

$= 4.94 \text{ [J]}$

$E_A = \frac{1}{2} k A^2 = 4.94$

$\Rightarrow A = \sqrt{\frac{2 \cdot 4.94}{k}}$

$= 0.84 \text{ [m]}$

(c) distance left = amplitude − position

$= 0.84 - 0.60$

$= 0.24 \text{[m]}$

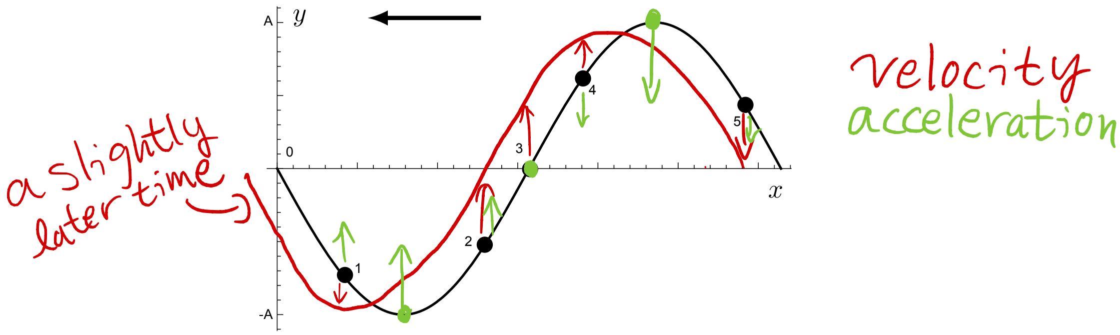
Problem 4. (20 Points)

A transverse wave is propagating in the $-x$ direction on a string at a speed of 2 [m/s], and a frequency of 10 [Hz]. At $t = 0$, the displacement at $x = 0$ is at its maximum negative value $A = 0.2$ [m].

(a) Write down the wave function. (10 points)

(b) Plot $v_y(x, t)$ at $t = 0$. (5 points)

(c) At a certain time, $t > 0$, the wave looks like the one in the figure below. Draw the direction of the velocity and acceleration at points 1, 2, 3, 4 and 5. (5 points)



(a) Generally $y(x,t) = A\cos(kx \pm \omega t)$

here, $A = -0.2$

$K = \frac{2\pi}{\lambda} = \frac{v}{f} = \frac{4\pi}{10} = 1.26$

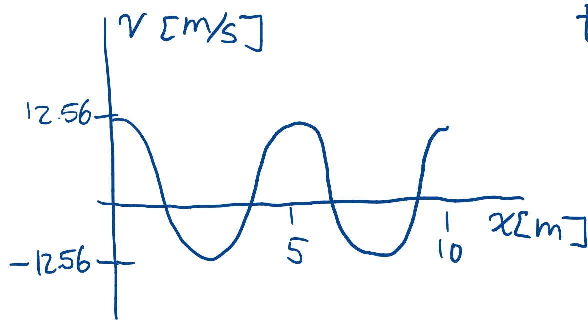
$\pm \rightarrow +$ (moving left)

$\omega = 2\pi f = 62.8$

So:

$$y(x,t) = -0.2\cos(1.26x + 62.8t)$$

(b) $V_y = \frac{\partial y}{\partial t} = 12.56\cos(1.26x \underbrace{+ 62.8t}_{t=0})$



(c) See above

Problem 5. (20 Points)

You are designing an instrument with two metal strings 35 [cm] long. Both strings are under the same tension. S_1 has a mass of 7 [g] and produces middle C in its fundamental mode.

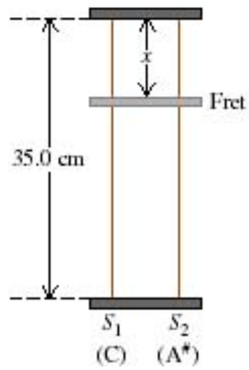
(a) Plot the first three normal modes of S_1 and label the nodal point(s) as N and antinodal points as A. (5 points)

(b) What is the tension on S_1? (5 points)

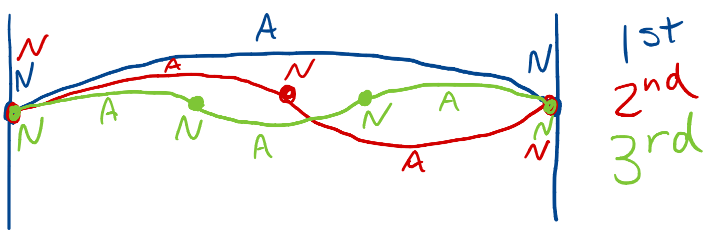
(c) What is the mass of S_2 if it produces A$^\sharp$ as its second harmonic? (5 points)

(d) To extend the range of your instrument, you include a fret located just under the strings, but not normally touching them. What is the x so that when you press S_1 tightly against it, this string will produce C$^\sharp$ as its second overtone?

Note: C $= 262$ [Hz], C$^\sharp = 277$ [Hz], and A$^\sharp = 466$ [Hz].



(a)



(b) Recall $v = \lambda f$ and $v = \sqrt{\frac{T}{\mu}}$

$$\lambda f = \sqrt{\frac{T}{\mu}} \Rightarrow T = \frac{m}{l}(\lambda f)^2 \quad , \text{ here } \lambda = 2l, \; f = 262$$

$\mu = m/l$

$$= 673 \text{ [N]}$$

(c) Second harmonic $\rightarrow \lambda = l$, $f = 466$ [Hz]

$$m = \frac{Tl}{(\lambda f)^2} = 8.85 \text{[g]}$$

(d) Second overtone $\rightarrow \lambda = \frac{2}{3}(l - x)$, $f = 277$ [Hz]

$$\lambda = \sqrt{\frac{Tl}{mf^2}} = \frac{2}{3}(l - x)$$

$$\Rightarrow x = l - \frac{3}{2}\sqrt{\frac{Tl}{mf^2}} = 0.0359 \text{ [m]}$$