Practice Final B
Solutions

(1) (a) $\lambda = \frac{v}{\omega} = \frac{v}{2\pi f} = \frac{3\times10^8}{7.10\times10^5 \cdot 2\pi} = 67.2\,[m]$

(b) $I = \frac{P}{A} = \frac{5\times10^4}{4\pi\cdot(1.77\times10^4)^2} = 1.27\times10^{-5}\left[\frac{W}{m^2}\right]$

(2) we determine:

$$\left.\begin{array}{l} K(0) = \frac{1}{2}\,1\cdot1^2 = \frac{1}{2} \\ U(0) = \frac{1}{2}\,1\cdot1^2 = \frac{1}{2} \end{array}\right\} \Rightarrow$$ 1/4 of the way through half a cycle

↳ $\phi = \frac{\pi}{4}$

Amplitude is where $K=0 \Rightarrow U=1 \Rightarrow A=\sqrt{2}$

$\omega = \sqrt{\frac{k}{m}} = 1$

Thus: $x(t) = A\sin(\omega t+\phi) = \sqrt{2}\sin(t+\pi/4)$

And: $U(t) = \frac{1}{2}kx^2 = 1\sin^2(t+\pi/4)$

(3)
- Traveling
- Most likely traveling
- Standing
- Standing
- Traveling

(4) The speed of sound in water is faster than in air.

$v = \sqrt{B/\rho} = \sqrt{\frac{2.2\times10^9}{10^3}} = 1483\left[\frac{m}{s}\right] > 343\left[\frac{m}{s}\right]$

(5) $A = 1.5$

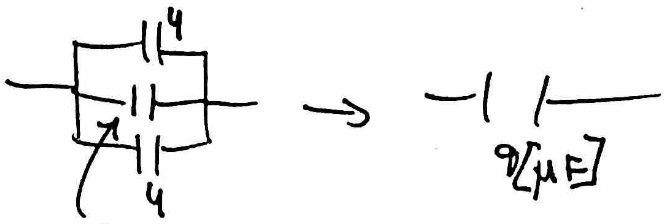
$k = 2$

$\pm = -$ } Right traveling

$\omega = 2$ } kinda hard to pinpoint so anything $1<\omega<3$ is full credit

$\phi = 0$

(6) (a)



$$\frac{1}{\frac{1}{2}+\frac{1}{2}} = 1$$

(b)

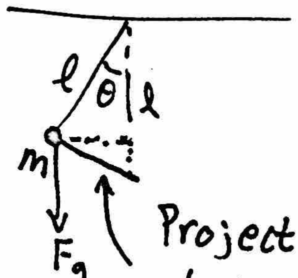
$$\frac{1}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{4}{3}[\Omega]$$

(c) Springs act like capacitors, so

$$9\left[\frac{N}{m}\right]$$

(7) For small angles θ,

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$



This is a SHO equation if $\sin\theta \sim \theta$, thus:

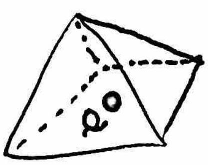
Project F_g here to find tangential acceleration $\ddot{\theta}$, − sign refers to restoring force

$$\ddot{\theta} = -\frac{g}{\ell}\theta = -\omega^2\theta$$

$$\Rightarrow T = 2\pi\sqrt{\frac{\ell}{g}} = \frac{2\pi}{\omega}$$

(8) $P = P_0 + \rho g h = 9.3\times10^6 + 730\cdot 8.87\cdot 15 = 9.40\times10^6\,[Pa]$

(9)

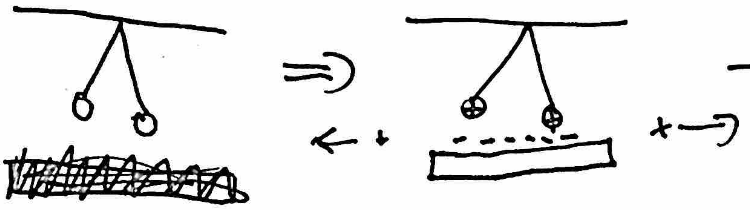


Half the flux goes below the pyramid, and so $\frac{1}{8}$th goes through each upper face.

$$\Phi = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \Phi_{one\,face} = \frac{Q_{enc}}{8\varepsilon_0}$$

(10) The larger sphere, they each have equal potentials at their surface

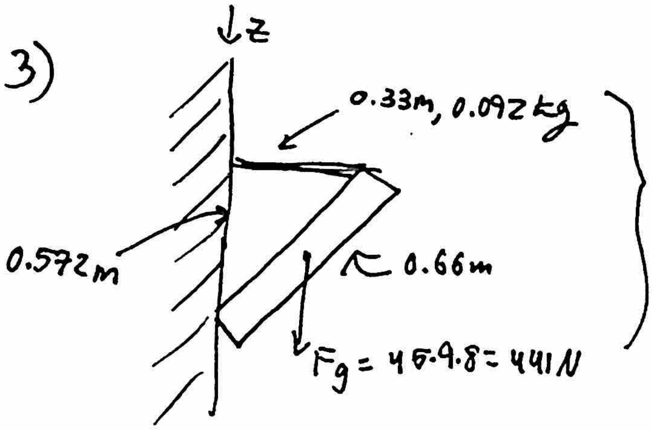
(11)



→ will be closer together if the plate is large since positive charges are pushed away and negative charges are attracted

(12) E : an infinite sheet is scale invariant!

(13)



Now, $F_z = 441 \cdot 0.33 \sin(\frac{\pi}{6}) \Rightarrow 127.3[N] = F$

$\mu = \frac{0.092}{0.33} = 0.276$

$v = \sqrt{\frac{F}{\mu}}, \quad f = \frac{v}{\lambda}$

$\Rightarrow f = 32.4\ Hz$

(1559 in your textbook)

(14) Treating these as point charges:

$$V = 2 \cdot \frac{q}{4\pi\varepsilon r} = \frac{2 \cdot 1.6\times10^{-19}}{4\pi \cdot 60 \cdot 8.8\times10^{-12} \cdot 10^{-9}} = 0.048[V]$$

$$C = \frac{Q}{V} = \frac{2 \cdot 1.609\times10^{-19}}{0.048} = 6.68\times10^{-18}[F]$$

$\Rightarrow$ Pair $\xRightarrow{2\times}$ $1.33\times10^{-17}[F]$

$\Rightarrow$ whole genome:

$$C = 3\times10^{9} \cdot 1.33\times10^{-17} = 4\times10^{-8}[F] = 40[nF]$$

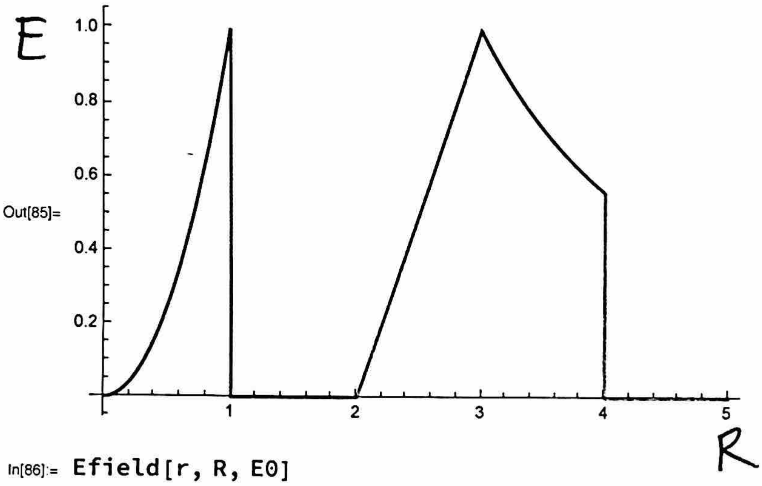
(15)

```
In[84]:= Efield[r_, R_, E0_] := Piecewise[{{E0 (r/R)^2, 0 ≤ r ≤ R},
      {0, R < r ≤ 2 R},
      {E0 (r/R - 2), 2 R < r ≤ 3 R},
      {E0 (3 R/r)^2, 3 R < r ≤ 4 R},
      {0, 4 R < r}}]
```

← Given this

```
Plot[Efield[r, 1, 1], {r, 0, 5}, Exclusions → None]
```

Out[85]=




```
In[86]:= Efield[r, R, E0]
```

Out[86]=

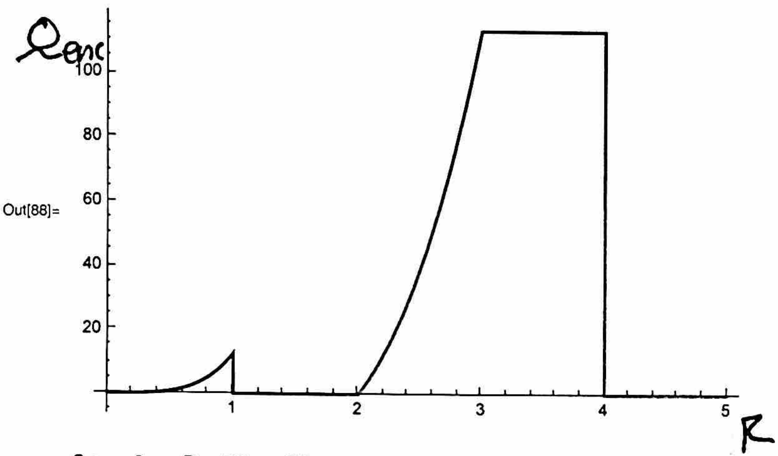
$$\begin{cases} \frac{E0\, r^2}{R^2} & 0 \le r \le R \\ 0 & R < r \le 2R \\ E0\left(-2 + \frac{r}{R}\right) & 2R < r \le 3R \\ \frac{9\, E0\, R^2}{r^2} & 3R < r \le 4R \\ 0 & \text{True} \end{cases}$$

$$Q_{enc} = 4\pi\varepsilon_0 r^2 E$$

from spherical symetry

In[87]:=
```
Qenc[r_, R_, E0_, ϵ0_] := Piecewise[{{4 π ϵ0 r^2 E0 (r/R)^2, 0 ≤ r ≤ R},
    {0, R < r ≤ 2 R},
    {4 π ϵ0 r^2 E0 (r/R - 2), 2 R < r ≤ 3 R},
    {4 π ϵ0 r^2 E0 (3 R/r)^2, 3 R < r ≤ 4 R},
    {0, 4 R < r}}]
  Plot[Qenc[r, 1, 1, 1], {r, 0, 5}, Exclusions → None]
```

Out[88]=



In[89]:=
```
Qenc[r, R, E0, ϵ0]
```

Out[89]=

$$\begin{cases} \frac{4\,E0\,\pi\,r^4\,\epsilon 0}{R^2} & 0 \le r \le R \\ 0 & R < r \le 2R \\ 4\,E0\,\pi\,r^2\left(-2 + \frac{r}{R}\right)\epsilon 0 & 2R < r \le 3R \\ 36\,E0\,\pi\,R^2\,\epsilon 0 & 3R < r \le 4R \\ 0 & \text{True} \end{cases}$$

In[90]:=

$$D\left[\left(\frac{r}{R}\right)^2, r\right]$$

$$D\left[\left(\frac{r}{R} - 2\right), r\right]$$

$$dQ = \rho\, dV = \rho \cdot 4\pi r^2 dr$$

$$D\left[\left(\frac{3R}{r}\right)^2, r\right]$$

$$\Rightarrow \rho = \frac{1}{4\pi r^2}\frac{dQ}{dr}$$

Out[90]= $\frac{2r}{R^2}$

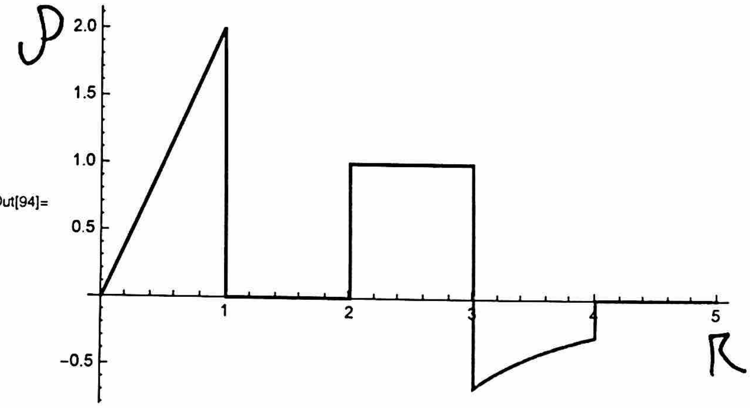
Out[91]= $\frac{1}{R}$

Out[92]= $-\frac{18R^2}{r^3}$

In[93]:=
```
rho[r_, R_, E0_, ϵ0_] := Piecewise[{{ϵ0 E0 (2 r)/R^2, 0 ≤ r ≤ R},
    {0, R < r ≤ 2 R},
    {ϵ0 E0 1/R, 2 R < r ≤ 3 R},
    {-ϵ0 E0 (18 R^2)/r^3, 3 R < r ≤ 4 R},
    {0, 4 R < r}}]
  Plot[rho[r, 1, 1, 1], {r, 0, 5}, Exclusions → None]
```

← Substitute in

Out[94]=



In[95]:=
```
rho[r, R, E0, ϵ0]
```

Out[95]=

$$\begin{cases} \frac{2\, E0\, r\, \epsilon0}{R^2} & 0 \le r \le R \\ 0 & R < r \le 2R \\ \frac{E0\, \epsilon0}{R} & 2R < r \le 3R \\ -\frac{18\, E0\, R^2\, \epsilon0}{r^3} & 3R < r \le 4R \\ 0 & \text{True} \end{cases}$$

we know this because E is discontinuous here

In[96]:=

$$-\frac{1}{4\pi R^2}\,\frac{4\, E0\, \pi R^4\, \epsilon0}{R^2}$$ ← Surface charge at $r=R$

$$-\frac{1}{4\pi (4R)^2}\, 36\, E0\, \pi R^2\, \epsilon0$$ ← Surface charge at $r=4R$

Out[96]= $-E0\, \epsilon0$ $= \sigma(R)$

Out[97]= $-\frac{9\, E0\, \epsilon0}{16}$ $= \sigma(4R)$

```
In[119]:= $Assumptions = {Element[R, Reals]};
Efieldold[r_, R_, E0_] := Piecewise[{{E0 (r/R)^2, 0 ≤ r ≤ R},
   {0, R < r ≤ 2 R},
   {E0 (r/R - 2), 2 R < r ≤ 3 R},
   {E0 (3 R/r)^2, 3 R < r ≤ 4 R},
   {0, 4 R < r}}]
Efieldnew[r_, R_, E0_] := Piecewise[{{E0 (r/R)^2, 0 ≤ r ≤ R},
   {0, R < r ≤ 2 R},
   {E0 (r/R - 2), 2 R < r ≤ 3 R},
   {E0 (3 R/r)^2, 3 R < r ≤ 7 R / 2},
   {0, 7 R / 2 < r}}]
U = Integrate[ϵ0 Efieldold[r, R, E0]^2 / 2, {r, 0, ∞}][[1]][[1]][[1]]
U // N
U = Integrate[ϵ0 Efieldnew[r, R, E0]^2 / 2, {r, 0, ∞}][[1]][[1]][[1]]
U // N
```

Given

Out[122]= $\frac{1067\, E0^2\, R\, \epsilon 0}{1920}$

Out[123]= $0.555729\, E0^2\, R\, \epsilon 0$

Out[124]= $\frac{4649\, E0^2\, R\, \epsilon 0}{10290}$

Out[125]= $0.451798\, E0^2\, R\, \epsilon 0$

$U = \frac{\varepsilon_0}{2}\int_{\text{all Space}} E^2\, dV$

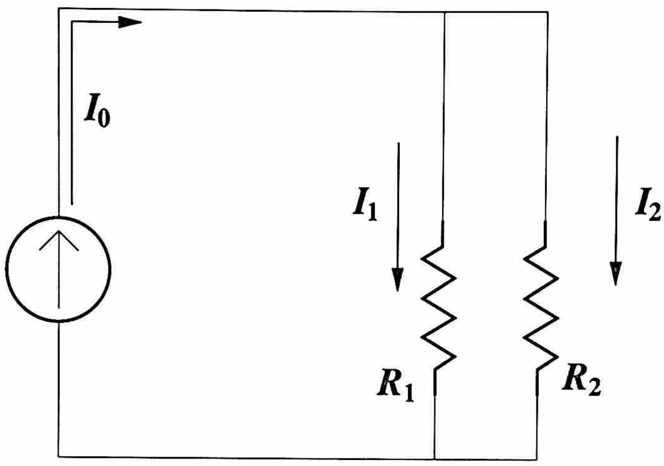
Intuitively, there is less of a polarization so less energy is needed in the second configuration. I.e. the system does work.

From MIT OCW 8.022

PROBLEM ~~[illegible]~~ (16)

In this problem, you will look at a circuit that contains a constant *current* source: the total current which comes out of this device is **always** I_0 no matter what EMF is required. The switch S is closed at $t = 0$.

First, this source is hooked up to a pair of resistors in parallel:



(a) Find the EMF $\mathcal{E}$ produced by the constant current source, as well as the currents I_1 (flowing through resistor R_1) and I_2 (flowing through R_2).

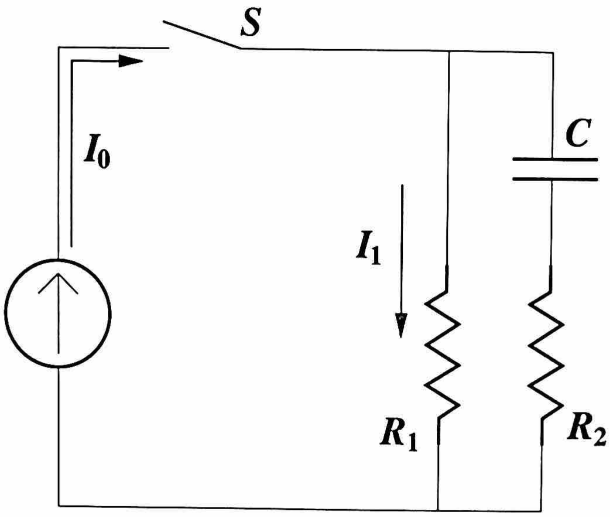
The resistors in parallel have an equivalent resistance given by $1/R_{eq} = 1/R_1 + 1/R_2$, or $R_{eq} = R_1R_2/(R_1 + R_2)$. The EMF produced by the current source is thus

$$\mathcal{E} = \frac{I_0 R_1 R_2}{R_1 + R_2} .$$

The currents through the two resistors are given by

$$\begin{aligned} I_1 &= \frac{\mathcal{E}}{R_1} = \frac{I_0 R_2}{R_1 + R_2} \\ I_2 &= \frac{\mathcal{E}}{R_2} = \frac{I_0 R_1}{R_1 + R_2} . \end{aligned}$$

The circuit is now modified: a capacitor and a switch are added. The capacitor is initially uncharged.



The switch is closed at $t = 0$.

(b) Find the initial currents $I_1(t = 0)$, $I_2(t = 0)$, and the late time currents, $I_1(t \to \infty)$, $I_2(t \to \infty)$. You should be able to do this with very little calculation. (Express your answers in terms of I_0 and the parameters of the circuit.)
At $t = 0$, there is no charge on the capacitor and it behaves identically to the circuit in part (a). At late times, the capacitor is fully charged and no current flows through R_2.

$$I_1(t=0) = \frac{I_0 R_2}{R_1 + R_2}\,, \qquad I_2(t=0) = \frac{I_0 R_1}{R_1 + R_2}$$
$$I_1(t \to \infty) = I_0\,, \qquad I_2(t \to \infty) = 0\,.$$

(c) Use Kirchhoff's laws to write down two equations relating $I_1(t)$, $I_2(t)$, I_0, and the charge on the capacitor Q. Write down a third equation relating Q to I_2.

$$I_0 = I_1 + I_2\,, \qquad I_1 R_1 - I_2 R_2 - \frac{Q}{C} = 0\,, \qquad I_2 = \frac{dQ}{dt}\,.$$

(d) Using the results of (c), find the late time charge, $Q(t \to \infty)$. (Express your answer in terms of I_0 and the parameters of the circuit.)
For $t \to \infty$ we have $I_2 = 0$, $I_1 = I_0$. Plugging into Kirchhoff and solving for Q, we find

$$I_0 R_1 - \frac{Q}{C} = 0$$
$$\longrightarrow Q = I_0 R_1 C\,.$$

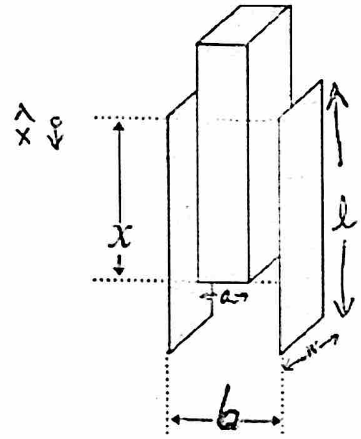
From MIT OCW 8.022

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Physics 8.022 *Spring 2003*

Final Exam

Problem (17) (Capacitance and Energy) (20 pts)



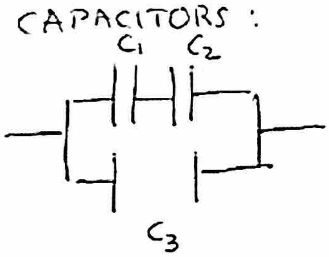
Consider a parallel plate capacitor with plate separation b and a sheet of metal of thickness a, as shown in the figure. Here, $b < a$. For this problem, consider the plates of the capacitor to be large enough so that fringing fields can be neglected.

a) Does the capacitance of the capacitor rise or fall after the metal sheet is inserted between the plates of the capacitor?

IT RISES (TOTAL CHARGE REMAINS SAME, VOLTAGE DROPS).

b) If the capacitance of the capacitor before the metal sheet was inserted was given by C_0, what is the capacitance after insertion of the metal sheet?

AT POSITION x THERE ARE EFFECTIVELY THREE CAPACITORS:



	Area	Length	C
C_1	$x \cdot w$	$\frac{b-a}{2}$	$\frac{xw}{4\pi(b-a)/2}$
C_2	xw	$\frac{b-a}{2}$	$\frac{xw}{4\pi(b-a)/2}$
C_3	$(\ell-x)\cdot w$	b	$\frac{(\ell-x)w}{4\pi b}$

$$C_{TOTAL} = C_3 + \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{(\ell-x)w}{4\pi b} + \frac{xw}{4\pi(b-a)} = \frac{\ell w}{4\pi b} + \frac{xwa}{4\pi b(b-a)}$$

$$= C_0 + \frac{xwa}{4\pi b(b-a)} = C_0\left[1 + \left(\frac{x}{\ell}\right)\left(\frac{a}{b-a}\right)\right]$$

ACCEPTABLE IF ASSUMED TOTALLY INSIDE, I.E $x=\ell \Rightarrow C_{TOTAL} = C_0 \frac{b}{b-a}$

c) Consider a charge an amount of charge $+Q$ and $-Q$ placed on the plates of the capacitor. What is the energy stored in the capacitor before and after the metal plate is inserted between the plates?

BEFORE $U_{BEFORE} = \frac{1}{2}\frac{Q^2}{C_{To}}$.

AFTER $U_{AFTER} = \frac{1}{2}\frac{Q^2}{C_{Total}} = \frac{1}{2}\frac{Q^2}{C_o}\frac{1}{1+\frac{a}{\ell(b-a)}x}$

↳ function of $\underline{x}$ Note: $1+\frac{ax}{\ell(b-a)} > 1 \Rightarrow U_{AFTER} < U_{BEF}$

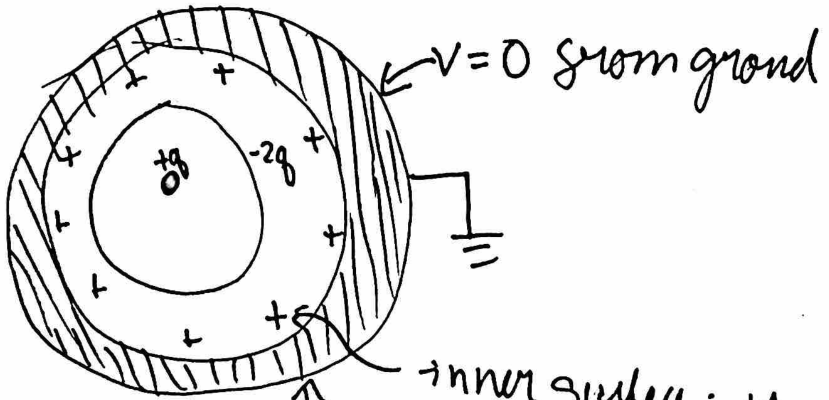
d) Given the equation $F = -\frac{\partial U}{\partial x}$, where x is the length of the metal plate that is inserted between the capacitor plates, what is the force on the metal? Which direction does it tend to move the metal plate?

$$F = -\frac{\partial U}{\partial x} = \frac{1}{2}\frac{Q^2}{C_o}\frac{1}{\left[1+\frac{ax}{\ell(b-a)}\right]^2}\cdot\frac{a}{\ell(b-a)}$$

The force is along $+\hat{x}$ (positive F resulted from derivative).

It pulls the metal sheet inside the capacitor

(18)



$$E = \frac{q_{enc}}{\varepsilon_0 a} = \frac{q_{enc}}{4\pi\varepsilon_0 r^2}$$

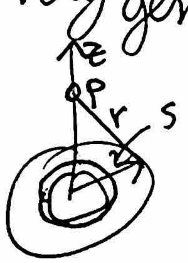
$$E = \begin{cases} \frac{q}{4\pi\varepsilon_0 r^2} & 0<r<r_1 \\ -\frac{q}{4\pi\varepsilon_0 r^2} & r_1<r<r_2 \\ 0 & r_2<r<r_3 \\ 0 & r_3<r<\infty \end{cases}$$

(19) Use Bernoullis Equation:

$$\cancel{P_i} + \underbrace{\rho}_{10^3} \underbrace{g}_{9.8} \underbrace{y_i}_{96} + \frac{1}{2}\underbrace{\rho}_{10^3} \underbrace{v_i^2}_{1} = \cancel{P_f} + \underbrace{\rho}_{10^3} \underbrace{g}_{9.8} \underbrace{y_f}_{0} + \frac{1}{2}\underbrace{\rho}_{10^3} \underbrace{v_f^2}_{?}$$

$\Rightarrow v_f = 43.4\left[\frac{m}{s}\right]$ (which is the same as if it fell 96 m starting at $v = 1\,m/s$)

(20) Very generic problem



$dQ(s) = 2\pi s\sigma\, ds$, $r = \sqrt{s^2+z^2}$

$$E_z = E\cdot\hat{z} = \int_0^a \frac{k\,dQ}{r^2}\,\hat{r}\cdot\hat{z} = \left(\int_0^a \frac{2\pi s\sigma\cdot k}{(s^2+z^2)} \frac{z}{(s^2+z^2)^{\frac{1}{2}}}\, ds\right)\hat{z}$$

integral you'd be given

$$= 2\pi\sigma k\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)$$