Practice Final A
Solutions

(1) Fastest speed is when all energy is kinetic, which is when the block passes equilibrium, like at $t=0$, so 1.5 m/s

Largest displacement is when all energy is potential, so $\frac{1}{2}kx^2 = \frac{1}{2}mv^2$, where $k = m\omega^2$, thus

$$\omega^2 x^2 = v^2 \Rightarrow x = \frac{v}{\omega} = \frac{1.5}{0.3} = \text{[redacted]}5 \text{ m}$$

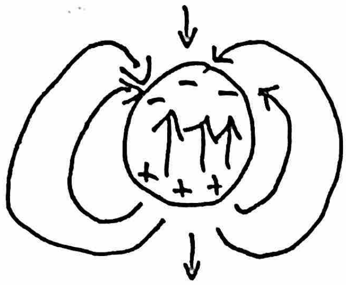
(2) $P = 4.71 \underbrace{\cos^2(2x - 3\pi t/2)}_{\text{max value is } +1}$

at $2x - \frac{3\pi}{2}t = n\pi \xRightarrow{x=0}$ at times $\frac{2n}{3}$ in seconds

$P_{max} = 4.71$ [W]

(3)
- longitudinal
- most likely neither, but possibly visualized as longitudinal
- transverse
- transverse
- longitudinal

(4)



(5)

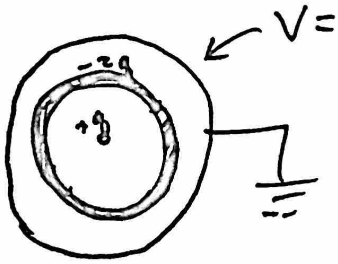
$$\omega_d = \frac{120}{60}\cdot 2\pi = 4\pi$$

$$F_d = mg = 40\cdot 60\cdot 10 = 2400\,[N]$$

$$A = \frac{2400}{\sqrt{(10^6 - 10^5(4\pi)^2)^2 + (4\pi\cdot 10^4)^2}} = 0.000162\,[m],$$

so this bridge is well designed to handle this load

(6)



$V=0 \Rightarrow q_{enc}=0 \Rightarrow q_{surface}=+q$

$$E = \frac{kq_{enc}}{r^2} \Rightarrow E = \begin{cases} \frac{kq}{r^2}, & r<r_1 \\ -\frac{kq}{r^2}, & r_2<r<r_3 \\ 0, & r_3<r \end{cases}$$

Note we avoided being asked for E in $r_1<r, r_2$ to save you time.

(7) (a) $\frac{P}{A} = I \Rightarrow P = IA = 1\frac{W}{m^2}\cdot 4\pi\cdot 3^2 = 36\pi\,[W]$

(b) $I = \frac{P}{A} = \frac{36\pi}{4\pi 1^2} = 4\ \frac{W}{m^2}$

(8) Mathematically, the position for stable equilibrium at x^* is that the electric potential $V(x^*)$ be at a <u>local</u> minimum. There, $\partial_{\hat{n}} V = 0 \Leftrightarrow E = 0 \Leftrightarrow F = 0$,

and $\partial^2_{\hat{n}\hat{n}} V > 0 \Rightarrow$ stable (restoring force)

where $\hat{n}$ can be any direction

Gauss's law probably doesn't help you.

(9) $F = PA = 1.01\times10^5 \cdot 4\pi \cdot 0.1^2 = 4.04\pi\times10^3\,[N] \approx 2800$ pounds

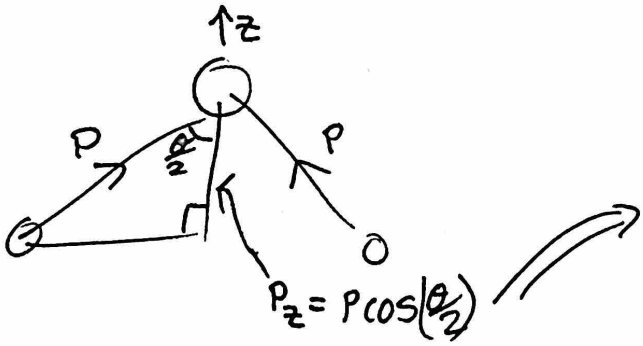
"you have a <u>ton</u> of pressure on you"

$\bar{P}_{\text{bottom of pool}} = P_0 + \rho g h = 1.01\times10^5 + 10^3 \cdot 9.8 \cdot 3 = 1.30\times10^5\,[Pa]$

$F = PA = 1.30\times10^5 \cdot 4\pi \cdot 0.1^2 = 5.20\pi\times10^3\,[N] \approx 3600$ pounds

"Thus this exam should be relaxing since relative to being in the pool there's little pressure on you"

(10)

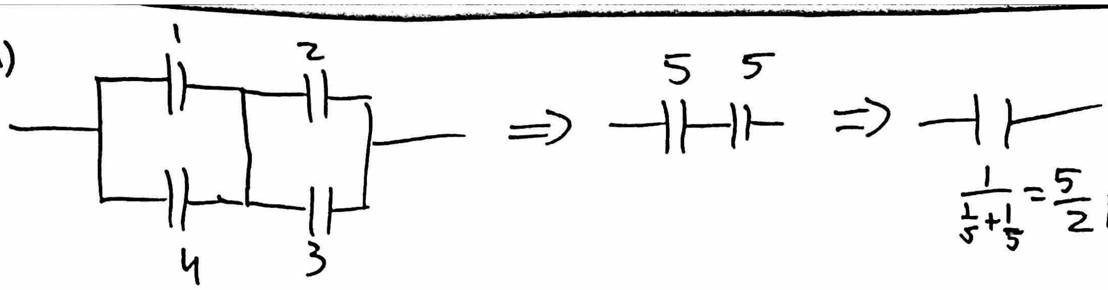


$P_{Net} = 2p\cos(\theta/2)\,\hat{z}$

$= 2ed\cos(\theta/2)\,\hat{z}$

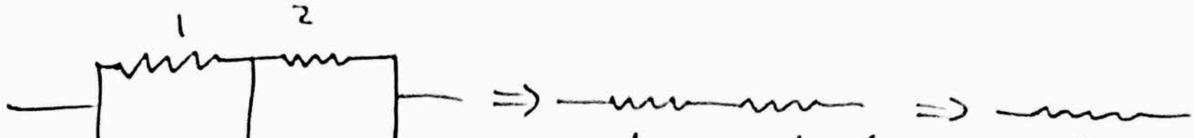
No net force from a symmetric/uniform electric field, while there would be if there were <u>dissolved ions</u>. If interested, see the Veritasium video.

(11) (a)



$$\frac{1}{\frac{1}{5}+\frac{1}{5}} = \frac{5}{2}\,\mu F$$

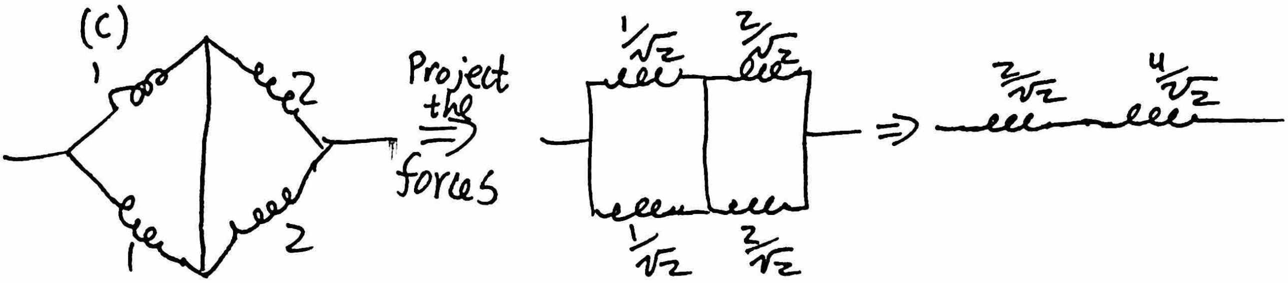
(b)



$$\frac{1}{\frac{1}{1}+\frac{1}{4}} = \frac{4}{5} \qquad \frac{1}{\frac{1}{2}+\frac{1}{3}} = \frac{6}{5}$$

$2\,\Omega$

(c)



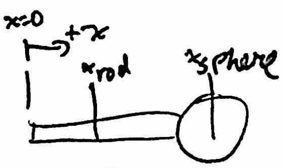
$$\Rightarrow \quad \frac{1}{\frac{1}{\frac{2}{\sqrt{2}}}+\frac{1}{\frac{4}{\sqrt{2}}}} = \frac{2\sqrt{2}}{3}\ \frac{N}{m}$$

(12)

$$I_{rod} = \frac{m(4a)^2}{3}$$

$$I_{disc} = \underbrace{I_{disc}}_{\frac{ma^2}{2}} + \underbrace{I_{Parallel}}_{m\cdot(5a)^2}$$

$$I_{total} = \frac{185\,ma^2}{6}$$



$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{185\,ma^2}{6mgd}} = 2\pi\sqrt{\frac{185a}{21g}} \approx 5.96\sqrt{a}$$

$g = 9.8$

$$d = \frac{7a}{2}$$

$$d = \frac{m\cdot 2a + m\cdot 5a}{2m} = \frac{7a}{2}$$

(13) (a) $R = \int dR = \int_0^l \frac{\rho\, dl}{A} = \int_0^l \frac{\rho_0 \sin(\pi x/l)}{\pi r^2}\, dx = \frac{2 \rho_0 l}{\pi^2 r^2}$

$\uparrow$

$\int_0^l \sin\left(\frac{\pi x}{l}\right) dx = \frac{2l}{\pi}$

(b) $R = \frac{2 \cdot 1 \cdot 0.2}{\pi^2 \cdot 0.01^2} = 405\ \Omega$

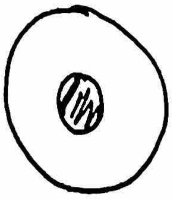
$\Rightarrow \mathcal{E} = IR \rightarrow \mathcal{E} = 5 \cdot 405 = 2025\ V \leftarrow$ "big"

(c) $P = IV = 5 \cdot 2025 = 10125\ W \approx 10\,kW$

(14)

(a) $[c_1] = \frac{C}{m^3}$

$[c_2] = \frac{C}{m^2}$



(b) $\vec{E} = \frac{q_{enc}}{\varepsilon_0 A}\hat{r}$ with spherical symmetry like this

$$q_{enc} = \int_0^r dr\, \rho \cdot 4\pi r^2 = \frac{4\pi c_1}{r_1}\int_0^r dr\, r^3 = \frac{\cancel{4}\pi c_1}{r_1} r^4$$

$dV = 4\pi r^2 dr$

$$\vec{E} = \frac{\cancel{\pi} c_1 \quad r^{\cancel{4}2}}{4\cancel{\pi}\varepsilon_0 r_1 \cancel{r^2}}\hat{r} = \frac{c_1}{4 r_1 \varepsilon_0} r^2 \hat{r} \qquad \text{for } 0<r<r_1$$

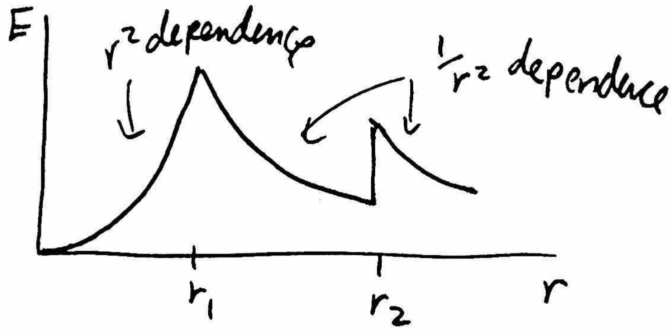
(c) $q_{enc} = \frac{\pi c_1}{r_1} r_1^4 = \pi c_1 r_1^3$

$$\vec{E} = \frac{\cancel{\pi} c_1 r_1^3}{4\cancel{\pi}\varepsilon_0 r^2}\hat{r} = \frac{c_1 r_1^3}{4\varepsilon_0}\cdot\frac{1}{r^2}\hat{r} \qquad \text{for } r_1<r<r_2$$

(d) $q_{enc} = \pi c_1 r_1^3 + 4\pi r_2^2 \cdot c_2 \cdot \left(\frac{r_2}{r_2}\right)^2 = \pi c_1 r_1^3 + 4\pi c_2 r_2^2$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^2}\cdot(\pi c_1 r_1^3 + 4\pi c_2 r_2^2)\hat{r} \qquad \text{for } r_2<r$$

(e)



It is ok for E ~~to~~ be discontinuous when there is a surface charge.

(f) $V(r)$ for $r_2<r$:

$$V(r) = \int_r^\infty E\cdot d\vec{r} = \frac{\pi c_1 r_1^3 + 4\pi c_2 r_2^2}{4\pi\varepsilon_0}\int_r^\infty \frac{1}{r^2}dr = \frac{\pi c_1 r_1^3 + 4\pi c_2 r_2^2}{4\pi\varepsilon_0}\,\frac{1}{r}$$

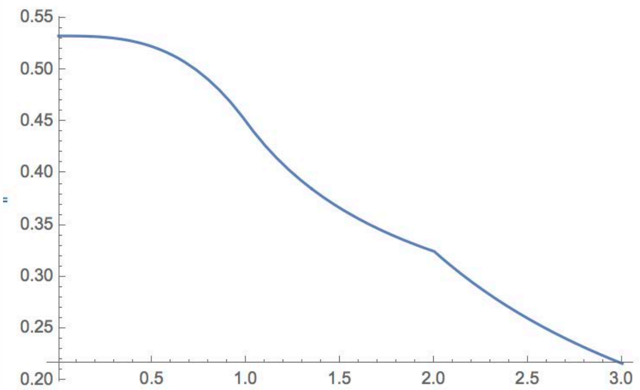
for $r_1<r<r_2$: $V(r) = \int_{r}^\infty E\cdot d\vec{r} = \int_r^{r_2} E\cdot d\vec{r} + \int_{r_2}^\infty E\cdot d\vec{r} = -\frac{c_1 r_1^3}{4\varepsilon_0}\left(\frac{1}{r_2}-\frac{1}{r}\right) + \frac{\pi c_1 r_1^3 + 4\pi c_2 r_2^2}{4\pi\varepsilon_0}\,\frac{1}{r_2}$

$$= \frac{c_1 r_1^3}{4\varepsilon_0}\cdot\frac{1}{r} + \frac{c_2 r_2^2}{\varepsilon_0}\cdot\frac{1}{r_2}$$

for $0<r<r_1$: $V(r) = \int_r^{r_1} E \cdot dr + \underbrace{\int_{r_1}^{r_2} E \cdot dr + \int_{r_2}^{\infty} E \cdot dr}_{\frac{C_1 r_1^2}{4\varepsilon_0} + \frac{C_2 r_2}{\varepsilon_0}}$

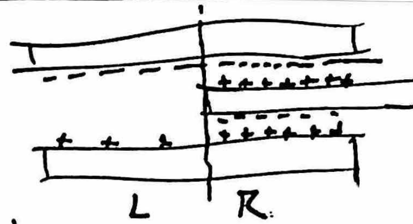
$$V(r) = \frac{C_1}{12 r_1 \varepsilon_0}\left(r_1^3 - r^3\right) + \frac{C_1 r_1^2}{4\varepsilon_0} + \frac{C_2 r_2}{\varepsilon_0}$$

Thus, we may plot:



Note that this is continuous as we desire

(15) The plates are conducting, so $\Delta V_L = \Delta V_R$,



and $\Delta V_L = E_L \cdot S$, while $\Delta V_R = E_R \frac{S}{2}$ ← because no $\vec{E}$ in metal.

Thus $E_R = 2E_L$, while $E_L = \frac{\sigma_L}{\varepsilon_0}$, while $E_R = \frac{\sigma_R}{\varepsilon_0}$

thus, $\sigma_R = 2\sigma_L$, meanwhile $\sigma = \frac{Q}{L^2} \Rightarrow \sigma_R = \frac{2Q}{3L^2}$; $\sigma_L = \frac{Q}{3L^2}$

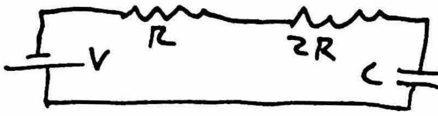
Which means: $E_R = \frac{\sigma_R}{\varepsilon_0} = \frac{2Q}{3\varepsilon_0 L^2}$, $E_L = \frac{\sigma_L}{\varepsilon_0} = \frac{Q}{3\varepsilon_0 L^2}$

$\Delta V_L = E_L \cdot S = \frac{QS}{3\varepsilon_0 L^2}$, $\Delta V_R = E_R \cdot \frac{S}{2} = \frac{Q}{3\varepsilon_0 L^2}$

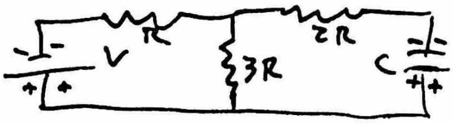
$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q}S}{3\varepsilon_0 L^2}} = \frac{3\varepsilon_0 L^2}{S}$$

$$U = CV^2 = \frac{3\varepsilon_0 L^2}{S} \cdot \left(\frac{Q^2 S^2}{3^2 \varepsilon_0^2 L^4}\right) = \frac{Q^2 S}{3\varepsilon_0 L^2}$$

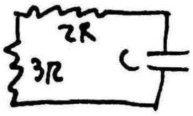
(16) For $t_0 < t < t_1$:



For $t_1 < t < t_2$:

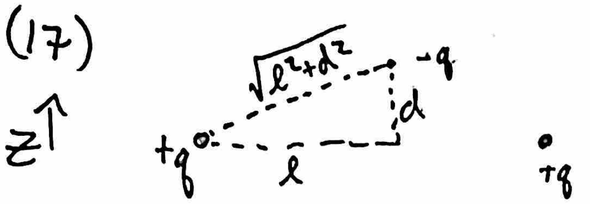


For $t_2 < t < \infty$:



See handout for solutions

(17)



In the limit $d \ll l$, $\sqrt{l^2+d^2} \approx l$, so

$$F = \underset{\text{restoring}}{-q}\,E \quad \text{where} \quad E = 2 \cdot \frac{kq}{\underset{l}{r^2}}\, \underbrace{\hat{r}\cdot\hat{z}}_{\frac{d}{l}}$$

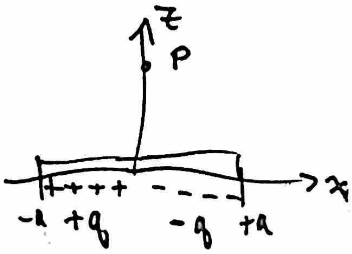
Thus: $F = -\dfrac{2kq^2}{l^3} d$ (like $F = -kx$)

$$F = ma = m\ddot{x} \longrightarrow \ddot{d} = -\underbrace{\frac{2kq^2}{ml^3}}_{\omega^2} d \implies \omega = \sqrt{\frac{2kq^2}{ml^3}}$$

(18) (a) $f_L = \dfrac{\overset{\text{speed of sound}}{340} - 0}{340-6} \cdot 300 = 305.4$ [Hz] heard by spectator

(b) $f_L = \dfrac{340+6}{340+0} \cdot 250 = 254.4$ [Hz] heard by runner

(19)



$$\sigma_L = \frac{q}{a}$$

$$\sigma_R = -\frac{q}{a}$$

Here we note that the $\hat{z}$ components cancel, but the x-components do not, so:

$$E_x(P) = \int \frac{k\,dq}{r^2} = 2\int_{-a}^{0} \frac{kq}{a} \cdot \frac{1}{\sqrt{x^2+z^2}}\,dx \cdot \underbrace{\frac{x}{z}}_{x\text{-component}} = \frac{2kq}{a}\int_{-a}^{0} dx \frac{x}{\sqrt{x^2+z^2}}$$

$\implies$ use integrals given $\implies E = \dfrac{2kq}{a} \dfrac{1}{(x^2+a^2)^{1/2}}\,\hat{x}$

(20) Use Bernoulli's Equation with an additional term:

$$\cancel{P_i} + \underbrace{\rho}_{10^3}\underbrace{g}_{9.8}\underbrace{y_i}_{258} + \cancel{\tfrac{1}{2}\rho \underbrace{v_i^2}_{0}} = \cancel{P_f} + \underbrace{\rho}_{10^3}\underbrace{g}_{9.8}\underbrace{y_f}_{96} + \frac{1}{2}\underbrace{\rho}_{10^3}\underbrace{v_f^2}_{5} + \frac{\text{energy}}{m^3}$$

$$\implies \frac{\text{energy}}{m^3} = 1.58\times10^6 \frac{J}{m^3} \implies 1.58\times10^6 \frac{J}{m^3} \cdot \underbrace{(10000\,\pi\, 0.005^2)\,5}_{\frac{m^3}{s}} = 6.21\times10^6 [W]$$