## Vectors

$\boldsymbol{v} = (v_x, v_y, v_z)$ is a vector with components $v_x$, $v_y$, and $v_z$. To add vectors we have $\boldsymbol{u}+\boldsymbol{v} = (u_x+v_x, u_y+v_y, u_z+v_z)$. To multiply we have the dot product

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_x v_x + u_y v_y + u_z v_z$$

or for angle $\theta$ between $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos(\theta)$$

where the length of the vector is

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The cross product is another multiply

$$\boldsymbol{u}\times\boldsymbol{v} = \det\begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are the unit vectors (length 1) in the $x$, $y$, and $z$ directions.

## Velocity and Acceleration

The velocity is

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$$

the momentum is

$$\boldsymbol{p} = m\boldsymbol{v}$$

and the acceleration is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{x}}{dt^2}$$

As an integral equation

$$\boldsymbol{x}(t) = x(0) + \int_0^t d\bar{t}\,\left(\boldsymbol{v}(0) + \int_0^{\bar{t}} d\bar{\bar{t}}\,\boldsymbol{a}(\bar{\bar{t}})\right)$$

## Forces and Newton's Laws

The total force is the sum of forces:

$$\boldsymbol{F}_{\text{tot}} = \sum_i \boldsymbol{F}_i = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \dots$$

Newton's 1st law: when $\boldsymbol{F}_{\text{tot}}=0$, $\boldsymbol{a}=0$.
Newton's 2nd law: $\boldsymbol{F}_{\text{tot}} = m\boldsymbol{a} = d\boldsymbol{p}/dt$.
Newton's 3rd law: $\boldsymbol{F}_{A\to B} = -\boldsymbol{F}_{A\leftarrow B}$.

## Forces

The weight is the force that emerges from the linearization of gravity

$$\boldsymbol{F}_g = m\boldsymbol{g}$$

The force of a spring is

$$\boldsymbol{F}(x) = -kx\hat{\boldsymbol{x}}$$

Friction is a force that opposes motion. It is either *static*, in which case it opposes a force up up to magnitude $|\boldsymbol{F}_s|$, or *dynamic*/kinetic which exhibits a constant force $-|\boldsymbol{F}_d|\hat{v}$.

## Energy

The kinetic energy is

$$E_k = \frac{1}{2}m|\boldsymbol{v}|^2 = \frac{|\boldsymbol{p}|^2}{2m}$$

Work $W$ is the change in kinetic energy

$$W = E_k(t_2) - E_k(t_1)$$

where

$$W = \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} d\boldsymbol{l}\cdot\boldsymbol{F}$$

So:

$$W_{\text{grav}}(y) = mgy$$

$$W_{\text{spr}}(x) = \frac{1}{2}kx^2$$

From this we note that for conservative forces (forces that conserve energy):

$$\boldsymbol{F} = -\boldsymbol{\nabla}W$$

The power is

$$P = \frac{dW}{dt}$$

which is

$$P = \boldsymbol{F}\cdot\boldsymbol{v}$$

## Momentum

The total momentum is conserved

$$\boldsymbol{p}_{\text{tot}} = \sum_i \boldsymbol{p}_i = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \dots$$

The change in momentum is known as the "impulse", where $\boldsymbol{F} = d\boldsymbol{p}/dt$

$$\boldsymbol{J} = \boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{t_1}^{t_2} d\bar{t}\;\boldsymbol{F}_{\text{tot}}$$

In a collision, momentum is conserved. In an elastic collision, kinetic energy is conserved, and in an inelastic collision, kinetic energy is lost.

## Center of Mass

The center of mass is

$$\boldsymbol{x}_{\text{com}} = \frac{\sum_i m_i\boldsymbol{x}_i}{\sum_i m_i} = \frac{\sum_i m_i\boldsymbol{x}_i}{m_{\text{tot}}}$$

Where we have

$$\boldsymbol{p}_{\text{tot}} = m_{\text{tot}}\boldsymbol{v}_{\text{com}}$$

and for the total external forces

$$\boldsymbol{F}_{\text{ext}} = m_{\text{tot}}\boldsymbol{a}_{\text{com}}$$

## Equilibrium

Equilibrium is when all forces and torques sum to zero. Equilibrium is stable (when the potential creating the force/torque is concave up—$d^2W/dx^2 > 0$), or unstable (when the potential creating the force/torque is concave down—$d^2W/dx^2 < 0$).

## Projectile Motion

For a projectile launched at $\theta$, subject to $\boldsymbol{F}_g = -mg\hat{\boldsymbol{y}}$ we have positions

$$x(t) = x(0) + v(0)\cos(\theta)t$$

$$y(t) = y(0) + v(0)\sin(\theta)t - \tfrac{1}{2}gt^2$$

where the velocities are

$$\boldsymbol{v}(t)\cdot\hat{\boldsymbol{x}} = \boldsymbol{v}(0)\cdot\hat{\boldsymbol{x}}\cos(\theta)$$

$$\boldsymbol{v}(t)\cdot\hat{\boldsymbol{y}} = \boldsymbol{v}(0)\cdot\hat{\boldsymbol{y}}\sin(\theta) - gt$$

## Angular Motion

Rotational motion is described by an angle $\boldsymbol{\theta}$ and an angular velocity

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}$$

This is then changed by a torque $\boldsymbol{\Gamma}$

$$\boldsymbol{\Gamma} = I\frac{d\boldsymbol{\omega}}{dt} = I\frac{d^2\boldsymbol{\theta}}{dt^2}$$

which is given by

$$\boldsymbol{\Gamma} = \boldsymbol{r}\times\boldsymbol{F}$$

where the moment of inertia is

$$I = \sum_i m_i|\boldsymbol{r}_i|^2$$

The parallel axis theorem gives $I$ for rotation distance $d$ away from the com

$$I(d) = I_{\text{com}} + m_{\text{tot}}d^2$$

In analogy to linear motion

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(0) + \int_0^t d\bar{t}\,\left(\boldsymbol{\omega}(0) + \int_0^{\bar{t}} d\bar{\bar{t}}\,\boldsymbol{\Gamma}(\bar{\bar{t}})\right)$$

The kinetic energy is then ($\boldsymbol{v} = R\boldsymbol{\omega}$)

$$E_k = \frac{1}{2}I|\boldsymbol{\omega}|^2$$

The power is $P = \boldsymbol{\Gamma}\cdot\boldsymbol{\omega}$ and work is

$$W = \int_{\theta_1}^{\theta_2} d\boldsymbol{\theta}\cdot\boldsymbol{\Gamma}$$

The angular momentum of a particle is

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$$

or for an extended object

$$\boldsymbol{L} = I\boldsymbol{\omega}$$

Whence we have

$$\boldsymbol{\Gamma}_{\text{tot}} = \frac{d\boldsymbol{L}_{\text{tot}}}{dt}$$

## Uniform Circular Motion

In uniform circular motion

$$|\boldsymbol{v}| = \frac{2\pi R}{T}$$

No tangential, only radial acceleration

$$\boldsymbol{a}\cdot\hat{\boldsymbol{r}} = \frac{|\boldsymbol{v}|^2}{R} = \frac{4\pi^2 R}{T^2}$$

## Gravitation

It turns out that $\boldsymbol{F} = m\boldsymbol{g}$ is only an approximation, which holds since the radius of the earth is nearly constant. More generically:

$$\boldsymbol{F}_g = G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}$$

where the generating potential is

$$W = -G\frac{m_1 m_2}{r}$$

So we see that

$$\boldsymbol{g} = \frac{G m_{\text{earth}}}{r_{\text{earth}}^2}\hat{\boldsymbol{r}}$$

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