Useful Mathematics

You can cancel out any 0 and factor out all constants. The trig function are:

$$\sin(x) = O/H$$
$$\cos(x) = A/H$$
$$\tan(x) = O/A$$

For all real x, $x^2 \geq 0$, so we invent i such that $i^2 = -1$. The Gaussian is

$$P(x) = \frac{1}{\sqrt{2\pi}\Delta} e^{-(x-x_0)^2/2\Delta^2}$$

Integrating in polar coordinates we find

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Differentiating under the integral sign

$$\int_{-\infty}^{\infty} dx\ x^{2n} e^{-ax^2} = \left(\frac{-d}{da}\right)^n \sqrt{\frac{\pi}{a}}$$

Integrating directly we find

$$\int_{-\infty}^{\infty} dx\ e^{-ax} = \frac{1}{a}$$

Integrating by parts is very useful

$$\int_a^b fg'dx = \int_a^b fdg = -\int_a^b gdf + fg\Big|_a^b$$

For example, it gives us the Γ function

$$\Gamma(n+1) = \int_{-\infty}^{\infty} dx\ x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

Decomposing into even/odd functions

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

We then have $\int_{-a}^{a} dx\ \mathrm{odd}(x) = 0$ and

$$\int_{-a}^{a} dx\ \mathrm{even}(x) = 2\int_0^a dx\ \mathrm{even}(x)$$

Completing the square can also help

$$3x^2 - 12x = 3((x-2)^2 - 4)$$

The chain rule is also essential

$$\frac{d}{dx}(x\sin(x^2)) = \sin(x^2) + 2x^2\cos(x^2)$$

The u-Substitution helps as well

$$\int_1^4 dx\ 6xe^{-x^2} = \int_1^{16} du\ 3e^{-u}$$

We can do a u-substitution like

$$P_u(u)du = P_x(x)dx$$
$$n(u)du = n(E)d(E)$$

Physical Constants

$$q_e = 1.602 \cdot 10^{-19}\ \mathrm{C}$$
$$m_e = 9.109 \cdot 10^{-31}\ \mathrm{kg} = 511\ \mathrm{keV}$$
$$k_B = 8.617 \cdot 10^{-5}\ \mathrm{eV/K}$$
$$\hbar = 10.55 \cdot 10^{-34}\ \mathrm{J\ s}$$

Physics 1 Knowledge

The units of your answer should be sensible. Energy, momentum, and angular momentum are conserved. We have

$$\boldsymbol{F} = m\boldsymbol{a}$$

The work is

$$W = \int \boldsymbol{F}\cdot d\boldsymbol{l}$$

The kinetic energy is

$$K = \frac{1}{2}m|\boldsymbol{v}|^2 = \frac{|\boldsymbol{p}|^2}{2m}$$

and the Hamiltonian is

$$H = K + U$$

For a spring-mass system

$$\boldsymbol{F} = k\boldsymbol{x} \implies \omega = \sqrt{k/m} = 2\pi f$$

For relativistic motion we have

$$K = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2$$
$$pc = \sqrt{E^2 - m_0^2c^4} = \sqrt{K^2 + 2Km_0c^2}$$

The Lorentz force law is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

In a capacitor we have

$$|\boldsymbol{E}| = V/d$$

For a wave we have

$$f = c/\lambda$$

Waves can be added, leading to beats

$$\psi_{\text{total}}(x,t) = \psi_1(x,y) + \psi_2(x,t)$$

Probability Distributions

The expectation value of A is

$$\langle A\rangle = \int_{-\infty}^{\infty} dx\ P(x)A(x)$$

In radial coordinates this is

$$\langle A\rangle = \int_0^{\infty} dr\ P(r)A(r)$$

The probability is normalized if

$$\int_V P(V)\,dV = 1$$

Physically, $\psi(x)$ meaningless, but $P(x) = \psi^*(x)\psi(x) = |\psi(x)|^2$ is meaningful. In radial coordinates we have $dP(r)/dr = |\psi|^2 A$, or $P(r) = r^2|R(r)|^2$. From this we insist that ψ be finite and

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

The "uncertainty" in a quantity is

$$\Delta q \equiv \sqrt{\langle q^2\rangle - \langle q\rangle^2}$$

Uncertainty (Cauchy-Schwarz)

$$\Delta p\Delta x \geq \hbar/2$$
$$\Delta E\Delta t \geq \hbar/2$$

Operators

We have the operators

$$\hat{x} = x = i\hbar\frac{\partial}{\partial p}$$

where $\hat{x}$ and $\hat{p}$ are canonical conjugates

$$\hat{p} = p = -i\hbar\frac{\partial}{\partial x}$$

The Hamiltonian is

$$\hat{H} = \hat{K} + \hat{U}$$

Which is composed of kinetic energy

$$\hat{K} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

and potential energy

$$\hat{U} = U(x)$$

The energy generates time translation

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Schrödinger Equation

The Schrödinger equation is

$$\hat{H}\psi = \hat{E}\psi$$

With $\hat{E} = i\hbar\partial_t$ we have

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

For E_n an eigenvalue of $\hat{H}$ we have the time-independent Schrödinger equation (solutions have $\psi_n(t) = \psi_n(0)e^{-iE_nt/\hbar}$)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2} + U(x)\psi_n(x) = E_n\psi_n(x)$$

Solving we have ($\alpha = \sqrt{2m(U-E_n)/\hbar}$)

$$\psi_n(x) = Ae^{-\alpha x} + Be^{\alpha x}$$

In terms of trig functions ($k = i\alpha$)

$$\psi_n(x) = C\cos(kx) + D\sin(kx)$$

This solution holds in regions where U is constant. The boundaries conditions are that ψ and its first derivative are continuous (except when U is infinite). We can superimpose solutions

$$|\psi_1 + \psi_2|^2 = (\psi_1+\psi_2)(\psi_1+\psi_2)^*$$

which results in time-dependent beats.

Infinite Square Well

We consider the potential

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{otherwise}\end{cases}$$

For which the energy eigenvalues are

$$E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}$$

Which correspond to eigenfunctions

$$\psi_n(x) = \sqrt{2/L}\sin(n\pi x/L)$$

Harmonic Oscillator

We consider the potential

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

For which the energy eigenvalues are

$$E_n = (n + \tfrac{1}{2})\hbar\omega$$

with eigenfunctions ($a = \sqrt{\hbar/m\omega}$)

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi a^2}}} e^{-x^2/2a^2} H_n(\frac{x}{a})$$

Where the Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

And the ground state wavefunction is

$$\psi_0(x) = \frac{1}{\sqrt{\sqrt{\pi a^2}}} e^{-x^2/2a^2}$$

3D Infinite Square Well

In 3D, $\partial_{xx}^2 \mapsto \nabla^2$ so that the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi_n(\boldsymbol{r}) + U(\boldsymbol{r})\psi_n(\boldsymbol{r}) = E_n\psi_n(\boldsymbol{r})$$

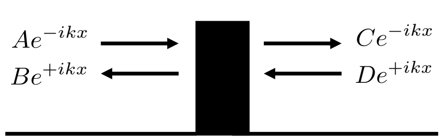
For a right parallelepiped we have

$$E_{ijk} = \frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_i}{L_1}\right)^2 + \left(\frac{n_j}{L_2}\right)^2 + \left(\frac{n_k}{L_3}\right)^2\right]$$

The linear momentum of the states is

$$|p_i| = n_i \frac{\pi\hbar}{L_i}$$

Quantum Tunneling



Reflection and transmission coefficients

$$\mathcal{R} \equiv \frac{|B|^2}{|A|^2}, \qquad \mathcal{T} \equiv \frac{|C|^2}{|A|^2}$$

Where there is no accumulation so

$$\mathcal{R} + \mathcal{T} = 1$$

We can approximate the transmission

$$\mathcal{T}(E) \approx e^{-\frac{2\sqrt{2m}}{\hbar}\int dx\ \sqrt{U(x)-E}}$$

Where we define the penetration depth

$$\delta = \hbar/\sqrt{2m(U-E)}$$

The energy levels of a finite well are

$$E_n \approx n^2 \frac{\pi^2\hbar^2}{2m(L+2\delta)^2}$$

Quantum Numbers

Quantum numbers are for counting to index eigenstates (usually nodes of the wavefunction). In hydrogenic atoms (and electronic orbitals in atoms)

- Principle, $n = 1, 2, 3, \ldots$
- Angular momentum, $l = 0, \ldots, n{-}1$
  - $l = 0 \leftrightarrow s,\ 1 \leftrightarrow p,\ 2 \leftrightarrow d,\ 3 \leftrightarrow f$
- Magnetic, $m_l = -l, \ldots, l-1, l$
- Spin, $s = -S, -S+1, \ldots, S$

For electrons and nucleons $S = 1/2$.

Ex. The $4F_{\frac{5}{2}}$ e^- is $n = 4$, $l = 3$, $j = \frac{5}{2}$.

Pauli exclusion principle: No two fermions may occupy an orbital with the same n, l, m_l, m_s numbers.

Angular Momentum

Where the total angular momentum is

$$j = l + s$$

Length of angular momentum vector

$$|\boldsymbol{J}| = \sqrt{j(j+1)}\hbar$$

The total orbital angular momentum

$$|\boldsymbol{L}| = \sqrt{l(l+1)}\hbar$$

The angular momentum along $\hat{\boldsymbol{z}}$ is

$$L_z = m_l\hbar$$

Which leads to "space quantization"

$$\cos(\theta) = \frac{L_z}{|\boldsymbol{L}|}$$

The spherical wave functions can be decomposed in the complete basis

$$\psi_{nlm_l}(r,\theta,\phi) = R_{nl}(r)Y_l^{m_l}(\theta,\phi)$$

Hund's rules for orbital occupation

1. Maximize the total spin
2. Maximize $\boldsymbol{J}$
3. Maximize $\boldsymbol{L}$

Wave-Particle Duality

Light has momentum

$$p_\gamma = \frac{h}{\lambda}$$

and energy

$$E_\gamma = hf = \frac{hc}{\lambda} = \frac{1240 \text{ eV nm}}{\lambda}$$

Which motivates wave-particle duality and the de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{hc}{pc}$$

Optical Evidence for Quantization

Wein Law approximation

$$\lambda_{\max} = \frac{0.002898}{T}$$

Plank's spectral radiance formula

$$u(f,T) = \frac{8\pi h f^3}{c^3}\left(e^{\frac{hf}{k_B T}} - 1\right)^{-1}$$

Energy quantization from a cavity

$$E = nhf = n\hbar\omega$$

Photoelectric effect with workfunction

$$K_{\max} = hf - \phi_{\text{WF}}$$

Electronic Evidence for Quantization

Bragg law for constructive interference

$$n\lambda = 2d\sin(\theta)$$

Compton scattering relation

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos(\theta))$$

Thompson's cathode ray experiment

$$\frac{e}{m_e} = \theta\left(\frac{V}{d}\right)\left(\frac{1}{B^2 l}\right)$$

Millikan oil drop experiment

$$ne = \frac{mg}{E}\left(\frac{v_{\text{ter}} + v_{\text{up}}}{v_{\text{ter}}}\right)$$

Hydrogen

Hydrogen has the Coulomb potential and energy levels ($k = 1/4\pi\epsilon_0$)

$$E_n = -\frac{ke^2}{2a_0}\frac{Z^2}{n^2} = -\frac{13.6Z^2}{n^2} \text{ eV}$$

which have negative energies and are bound states. Real eigenfunctions can always be found for bound states.

Stationary states of hydrogenic atoms

$$\psi(r,\theta,\phi,t) = R_{nl}(r)Y_l^{m_l}(\theta,\phi)e^{-i\omega t}$$

where R_{nl}, and $Y_l^{m_l}$ are tabulated.

The "radius" of wavefunctions is

$$r_n = \frac{a_0 n^2}{Z}$$

where a_0 is the Bohr radius

$$a_0 = \frac{\hbar^2}{m_e k e^2} = 0.0529 \text{ nm}$$

Quantum Combinatorics

Quantum effects onset at high density

$$\left(\frac{N}{L^{\text{dim}}}\right)\left(\frac{\hbar/2}{\sqrt{mk_BT}}\right)^{\text{dim}} \approx 1$$

Then we sum rather than integrate

$$E_{\text{tot}} = \sum_i n_i E_i$$

Where the number in in state i is

$$n_i(E_i) = g(E_i)f(E_i)$$

g_i is the density of states and f_i is the distribution function. In continuum

$$n(E)dE = g(E)f_{\text{dist}}(E)dE$$

Distribution Functions

Classical (distinguishable), ex. gasses

$$f_{\text{MB}}(E_i, T) = \left(e^{\frac{E_i}{k_BT}}\right)^{-1}$$

Bosons (S = integer, indistinguishable), ex. photons, phonons, gluons

$$f_{\text{BE}}(E_i, T) = \left(e^{\frac{E_i}{k_BT}} - 1\right)^{-1}$$

Fermions (S = integer + $\frac{1}{2}$, indistinguishable), ex. electrons, neutrinos

$$f_{\text{FD}}(E_i, T) = \left(e^{\frac{E_i}{k_BT}} + 1\right)^{-1}$$

Maxwell-Boltzmann Gas

The speed distribution of a gas is

$$n(v)dv = \frac{4\pi N}{V}\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} v^2 e^{-\frac{mv^2}{2k_BT}}dv$$

We can then find the mean using

$$\langle v\rangle = \frac{\int_0^\infty dv\ vn(v)}{\int_0^\infty dv\ n(v)}$$

or the root-mean-square (RMS) using

$$\sqrt{v^2} = \sqrt{\frac{\int_0^\infty dv\ v^2n(v)}{\int_0^\infty dv\ n(v)}}$$

or the extremal points using

$$\frac{d(n(v))}{dv} = 0$$

Blackbody Radiation

The spectra radiance is given by

$$u(E)dE = E\,n(E)dE = \frac{g_\gamma(E)EdE}{e^{\frac{E}{k_BT}} - 1}$$

The total number with frequency is

$$N(f)df = \frac{8\pi f^2 df}{c^3} = \frac{8\pi e^2 dE}{(hc)^3} = g_\gamma dE$$

So we have density of states

$$g_\gamma(E) = \frac{8\pi E^2}{(hc)^3}$$

Einstein Heat Capacity

The specific heat capacity is defined by

$$C \equiv \frac{dU}{dT}$$

Where for classical materials have

$$U = 3N_Ak_BT = 3RT$$

Which gives $C = 3R$, corresponding to one R per degree of freedom. If instead we model the energy in a solid as being sequestered solely in phonons then

$$\langle E\rangle = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_BT}} - 1}$$

Or for a macroscopic sample of solid

$$U = 3N_A\langle E\rangle$$

Which gives us the specific heat

$$C = 3R\left(\frac{\hbar\omega}{k_BT}\right)^2 \frac{e^{\hbar\omega/k_BT}}{(e^{\hbar\omega/k_BT} - 1)^2}$$

Where the Einstein temperature is

$$T_E = \frac{\hbar\omega}{k_B}$$

Free Electron Gas

For a free-electron gas we have

$$E = \frac{|\boldsymbol{p}|^2}{2m_e} = \frac{\hbar^2|\boldsymbol{k}|^2}{2m_e}$$

Which means that

$$d|\boldsymbol{k}| = \frac{1}{2}\left(\frac{2m_e}{\hbar^2}\right)^{\frac{1}{2}} E^{-\frac{1}{2}}dE$$

So the density of states is then

$$g(E) = \frac{8\sqrt{2}\pi m_e^{\frac{3}{2}}}{h^3}E^{\frac{1}{2}}$$

Which means the number at E is

$$n(E)dE = \frac{8\sqrt{2}\pi m_e^{\frac{3}{2}}}{h^3}\frac{E^{\frac{1}{2}}dE}{e^{(E-\mu)/k_BT} + 1}$$

So the chemical potential $T = 0$ is

$$E_F = \mu(0) = \frac{h^2}{2m_e}\left(\frac{3N}{8\pi V}\right)^{\frac{2}{3}}$$

The Fermi velocity is given by

$$\frac{1}{2}m_ev_F^2 = E_F$$

And we have the Fermi temperature

$$T_F \equiv \frac{E_F}{k_B}$$

And Fermi wavenumber

$$|\boldsymbol{k}_F| = \sqrt{2m_eE_F}/\hbar$$

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