Complex Analysis Notes

1.1 $z = x + iy$

$\underbrace{x}_{Re(z)} \quad \underbrace{y}_{Im(z)}$

modulus $|z| = \sqrt{x^2+y^2}$

Triangle Inequality $|z+w| \leq |z| + |w|$

Conjugate $\bar{z} = x - iy$ → reflection across x-axis

$$\boxed{\frac{1}{z} = \frac{\bar{z}}{|z|^2}}$$

$$\bar{\bar{z}} = z$$

$$\overline{z+w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$|z| = |\bar{z}|$$

$$|z|^2 = z\bar{z}$$

$$Re(z) = \frac{z+\bar{z}}{2}$$

$$Im(z) = \frac{z-\bar{z}}{2i}$$

Fundamental thm. of algebra → n roots to nth degree equation

1.2 $x = r\cos\theta$

$y = r\sin\theta$

$z = \underbrace{r}_{|z|} e^{i\theta} = r(\cos\theta + i\sin\theta)$ $\quad -\pi < \theta \leq \pi$ → hence the slit plane

"arg z" ← $\arg z = Arg(z) + 2\pi k, \ k \in \mathbb{Z}$

$$e^{i(\theta + 2\pi m)} = e^{i\theta}$$

identities:

$$e^{i(\theta + 2\pi k)} = e^{i\theta}$$

$$|e^{i\theta}| = 1$$

$$e^{i(\theta+\varphi)} = e^{i\theta} e^{i\varphi}$$

$$\arg(\bar{z}) = -\arg(z)$$

$$\arg(1/z) = -\arg(z)$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

1.2 (cont)

de moirres formula

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

n-th root

$z^n = \omega$

$\rightarrow \boxed{\omega = \rho e^{i\varphi}}$

$\rightarrow r^n e^{in\theta} = \rho e^{i\varphi}$

$\Rightarrow \underbrace{r^n = \rho}_{\text{\& take usual + roots}}, \; n\theta = \varphi + 2\pi k, \; k \in \mathbb{Z}$

centered on zero and equally spaced about a circle

$\rightarrow \omega = 1 \Rightarrow z_k = e^{2\pi i k/n}$

1.3

Stereographic projection

$\Rightarrow$ look at it if you need it

1.4

Squaring

$\omega = z^2 = r^2 e^{2i\theta}$

$|\omega| = |z|^2$

$\arg \omega = 2 \arg z$

Square root

$\omega = z^{1/2} \Rightarrow \sqrt{r}\, e^{i\theta/2}$

$\Rightarrow$ this is an inverse, so we draw a branch cut at $\mathbb{C} \setminus (-\infty, 0]$ $\rightsquigarrow$ "somewhat arbitrary

$\rightarrow \varphi \in (\pi, \pi) \Rightarrow \varphi/2 \in (-\pi/2, \pi/2)$

1.5

$$\boxed{e^z = e^x \cos y + i e^x \sin y}$$

identities

$e^z = e^x e^{iy}$

$|e^z| = e^x$

$\arg(e^z) = y$

$e^{z+2\pi i k} = e^z \quad k \in \mathbb{Z}$

$e^{z+w} = e^z e^w$

vertical lines → circles

horizontal lines → rays (from the origin

1.6

$$\boxed{\log(z) = \log|z| + i\,\mathrm{Arg}(z) + 2\pi i k} \quad k \in \mathbb{Z}$$

inverse of the exponential → principle value → $k=0$

1.7

$$\boxed{z^\alpha = e^{\alpha \log z}}$$

$$= e^{\alpha(\log|z| + i\,\mathrm{Arg}(z) + 2\pi i k)}$$

take a slit from $\mathbb{C} \setminus [0, \infty)$

1.8

$$\left.\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \ ; \quad \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{array}\right\} \theta \Leftrightarrow z$$

$\cos(-z) = \cos(z) \quad ; \quad \sin(-z) = -\sin(z)$

2π - periodic

$\cos^2 z + \sin^2 z = 1$

hyperbolic

$\cosh(z) = \dfrac{e^z + e^{-z}}{2} \quad ; \quad \sinh(z) = \dfrac{e^z - e^{-z}}{2}$

$2\pi i$ periodic, also even/odd

identities

$\cosh(iz) = \cos z \quad , \quad \cos(iz) = \cosh(z)$

$\sinh(iz) = i\sin z \quad , \quad \sin(iz) = i\sinh(z)$

$\sin z = \sin x \cosh y + i\cos x \sinh y$

$\cos z = \cos x \cosh y - i\sin x \sinh y$

2.3

Analytic: $f(z)$ is analytic on U if $f(z)$ is differentiable at each $z \in U$, and the complex derivative is continuous
↳ goursat's thm. helps

→ Products, sums, quotients of analytic functions are analytic

let $f(z) = u(x) + i\,v(y)$, then

$f(z)$ is analytic on a domain D <u>if and only if</u>:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann Equations

Thm. if $f(z)$ is analytic on D, and $f'(z) = 0$, then $f(z)$ is constant

Thm. if $f(z)$ is analytic and real-valued on D, then $f(z)$ is constant

2.4 There are often inverse functions, and they can be calculated

2.5 Laplace equation:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots \frac{\partial^2}{\partial x_n^2} \right) f = 0$$

f is harmonic if all 1^{st} & 2^{nd} order partial derivatives exist, are continuous & satisfy the Laplace Equation.

Thm: if $f = \underbrace{u + iv}_{\text{harmonic conjugates}}$ is analytic, u and v are harmonic

→ unique to a constant

2.6 functions are "conformal" if they preserve angles

Thm: if $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$ then $f(z)$ is conformal at z_0.

Level curves of u & v

3.1 Line integrals along a path γ

→ change parameters, ex. $x = r\cos\theta$, $y = r\sin\theta$

Green's Theorem:

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

3.2 Fundamental Thm. of calculus:

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Line integrals are independent of path if

$\int_\gamma P\,dx + Q\,dy = 0$ for any closed path γ on D

4.1 $dz = dx + i\,dy$

Ex. $\displaystyle\int_{|z|=1} \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i$

ML Estimate:

$$\left|\int_\gamma h(z)\,dz\right| \le \underbrace{M}_{\underbrace{|h(z)| \le M}_{\text{function value}}} L \qquad L = \underbrace{\int_\gamma |dz|}_{\text{Path length}}$$

4.2 $F(z)$ is a primitive for $f(z)$ iff $F'(z) = f(z)$

4.3 Cauchy thm:

if D is bonded with a piecewise smooth boundary, if $f(z)$ is analytic on $D \cup \partial D$,

$$\int_{\partial D} f(z)\,dz = 0$$

4.4 Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0}\,dz \quad \& \quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

4.5 Cauchy Estimates & the Liouville Thm.

$$|f^{(n)}(z_0)| \le \frac{n!}{\rho^n} M_C \qquad |f(z)| \le M$$

Let $f(z)$ be an analytic function on $\mathbb{C}$, then if $f(z)$ is bounded, $f(z)$ is constant.

5.1 $\sum_{k=0}^{\infty} a_k$ converges to s if $s_k \to s$ where $s_k = a_0 + \dots + a_k$

Comparison test: if $0 \le a_k \le r_k$ if $\sum_k r_k$ converges, then $\sum_k a_k$ converges

"converges absolutely" iff $\sum_k |a_k|$ converges

converges absolutely $\Rightarrow$ converges

5.2 f_j converges pointwise on E iff $\forall x \in E$, $\{f_j(x)\}$ converges

converges uniformly to f on E if $|f_j(x) - f(x)| \le \varepsilon_j$, $\forall x \in E$

Weierstrass M-test: let $M_k \ge 0$ and $\sum M_k$ converges, if $g_k(x)$ are defined on E such that $|g_k(x)| \le M_k$, $\forall x \in E$, then $\sum g_k(x)$ converges uniformly on E.

Thm: if $\{f_k(z)\}$ is a sequence of analytic functions that converges uniformly to f, then f is analytic.

5.3 Power Series

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k$$

Radius of convergence R, converges iff $|z-z_0| < R$

Thm: $\sum_k a_k z^k$ a power series with radius of convergence R,

then:

$$f(z) = \sum_k a_k z^k, \quad |z| < R$$

is analytic

• Thm: if $\left|\frac{a_k}{a_{k+1}}\right|$ has a limit as $k \to \infty$ - either finite, or $+\infty$

then the radius of convergence is $R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$

5.4 Any function analytic on a disk D, has a power series on D

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \quad |z-z_0| < \rho$$

moreover, $a_k = \frac{f^{(k)}(z_0)}{k!}$ where:

$$a_k = \frac{1}{2\pi i} \oint_{|w-z_0| = r} \frac{f(w)}{|w-z_0|^{k+1}} \, dw \quad \text{for } 0 < r < \rho$$

Note:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$$

5.5 Power series at ∞ $\rightarrow$ $f(z) \rightarrow g(1/z)$

5.6 power series are essentially polynomials (and linear)

5.7 Zeros: $f(z)$ has a zero of order N at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$$

and

$$f^{(N)}(z_0) \neq 0$$

Thm: if D is a domain, and $f(z)$ is an analytic fuction on D that is not identically zero, then the zeros of $f(z)$ are isolated.

6.1 Sps. $f(z)$ analytic for $\rho < |z - z_0| < \sigma$, then $f(z) = f_0(z) + f_1(z)$ where f_0 is analytic for $|z - z_0| < \sigma$ and f_1 is analytic for $\rho < |z - z_0|$. if $f_1(\infty) = 0$, this decomposition is unique.

- Annuluses between singularities
- Partial fractions & geometric series can help:

Pg 164 Note : $\dfrac{1}{1 - g(z)} = g(z)^0 + g(z)^1 + g(z)^2 + \ldots$

6.2 Isolated singularites | Not isolated

ex. $\frac{1}{\sin(z)}$; $\frac{1}{z}$

ex. $\sqrt{z}$; $\ln(z)$

6.2 Cont

Types of singularities:

0 negative terms — Removable singularity — Riemann's thm for removable singularities: if z_0 is an isolated singularity of $f(z)$, then $f(z)$ is bounded around z_0

N negative terms — Pole of order N — z_0 is a pole of order N of $f(z)$ iff $f(z) = \frac{g(z)}{(z-z_0)^N}$ for analytic $g(z)$

∞ negative terms — Essential Singularity — Casorati-Weierstrass Thm: if z_0 is an essential singularity of $f(z)$, then there exists a sequence, $\forall w_0$, $z_n \to z_0$ such that $f(z_n) \to w_0$

7.1 The residue of $f(z)$ is the coefficient a_{-1} of the laurent exp.
→ partial fraction decomp is one way to get this

Residue Thm.

Let $f(z)$ be analytic on $D \cup \partial D$ except at finitely many isolated singularities $\{z_j\}$, then

$$\int_{\partial D} f(z)\, dz = 2\pi i \sum_j \operatorname{Res}[f(z), z_j]$$

Calculating Residues:

(1) If $f(z)$ has a simple pole at z_0, $\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

(2) If $f(z)$ has a double pole at z_0, $\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$

(3) If $f(z)$ and $g(z)$ are analytic at z_0, and $g(z)$ has a simple zero at z_0, then
$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

(4) If $g(z)$ is analytic at z_0 and has a simple zero at z_0, then $\operatorname{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$

7.2 $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$

For the ratio of polynomials $P(x)$, $Q(x)$ s.t.

$\deg(Q(x)) \geq \deg(P(x)) + 2$, we have:

$$\int_{-\infty}^{+\infty} dx\, \frac{P(x)}{Q(x)} = 2\pi i \sum_j \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$$

isolated singularities in the upper half plane

Similar for ratio of polynomials times trig functions