Physics 115C at UCLA ◊ Formula Sheet (1 of 3)

White Dwarf Stars

White dwarf stars are stars that exhibit quantum phenomena at a cosmic scale. Begin with the gravitational energy:

$$E_{\text{gravity}} = -\frac{3}{5}\frac{GM^2}{R}$$

The energy from degeneracy pressure:

$$E_{\text{degeneracy}} = \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{m_e}\frac{N_e^{5/3}}{R^2}$$

Combining these we find a relation:
$E_{\text{total}} = E_{\text{degen}} + E_{\text{grav}} = A/R^2 - B/R$
By extremization, the preferred radius:

$$\partial_R E_{\text{total}} = 0 \text{ when } R = 2A/B \implies$$

$$\implies R = \frac{(9\pi^2)^{2/3}}{8}\frac{\hbar^2}{m_e}\frac{1}{GM^{1/3}m_n^{5/2}}$$

Perturbation Theory

Perturbation theory is a formal way to work from what we know about to what we are interested in. Specifically:

$$H = H_0 + H'$$

In particular, if $H' \to \lambda H'$, we find that:

$$\psi_n = \psi_n^0 + \lambda\psi_n^1 + \lambda^2\psi_n^2 + \dots$$

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \dots$$

Non-Degenerate P.T.

The first-order corrections to energy:

$$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$$

The first-order corrections to states:

$$\psi_n^1 = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{E_n^0 - E_m^0}\psi_m^0$$

The second-order corrections to energy:

$$E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$$

Degenerate P.T.

With degeneracy, in particular when $H\psi_1^0 = E_a\psi_a^0$; $H\psi_1^0 = E_b\psi_b^0$; $E_a = E_b$, we proceed with a matrix method:

$$H'_{ij} = \langle\psi_i^0|H'|\psi_j^0\rangle$$

Defining $\psi^0 = \alpha\psi_a^0 + \beta\psi_b^0$, we solve:

$$\begin{pmatrix} H'_{aa} & H'_{ab} \\ H'_{ba} & H'_{bb}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = E^1\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

Eigenvalues are the first-order energy corrections, and eigenvectors are the first-order state corrections. These eigenvectors are "good states," where $A\psi_a^0 = a\psi_a^0$ and $A\psi_b^0 = b\psi_b^0$ for $a \neq b$. If $H'_{ab} = 0$ the solution is non-degenerate.

Some Applications of P.T.

Delta function in infinite square well, unperturbed is $E_n^0 = (\pi^2\hbar^2/2ma^2)n^2$:

$$H' = \alpha\delta(x - a/2)$$

With 1D non-degenerate P.T. find:

$$E_n^1 = \begin{cases} 2\alpha/a & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

$$E_n^2 = \begin{cases} -2m(\alpha/\pi\hbar n)^2 & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

Gaussian perturbation on free particle with L-periodic boundary conditions:

$$H' = -V_0\exp(-x^2/a^2)$$

With degenerate perturbation theory:

$$E_\pm^1 = H'_{aa} \pm |H'_{ab}|$$

$$= -\frac{a\sqrt{\pi}V_0}{L}\left\{1 \mp \exp\left[-\left(\frac{2n\pi a}{L}\right)^2\right]\right\}$$

Mathematical aside:

$$\int_{-\infty}^{+\infty} dx\, e^{-(Ax^2+Bx)} = \sqrt{\frac{\pi}{A}}\exp\left(\frac{B^2}{4A}\right)$$

Cubical box with introduced potential:

$$H' = V_0\,\theta(a-x)\theta(a-y)\theta(a-z)$$

Noting that unperturbed energies are:

$$E_{\boldsymbol{n}}^0 = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

In the basis of $\{\psi_{112}, \psi_{121}, \psi_{211}\}$:

$$H' = \frac{V_0}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{16}{\pi^2} \\ 0 & \frac{16}{\pi^2} & 1\end{pmatrix};\ E_1^1 = \begin{Bmatrix} E_1^0 + \frac{V_0}{4} \\ E_1^0 + \frac{V_0}{4}(1 + \frac{16}{\pi^2}) \\ E_1^0 + \frac{V_0}{4}(1 - \frac{16}{\pi^2})\end{Bmatrix}$$

Weak-Field Zeeman Effect is splitting from a magnetic field in atoms like Hydrogen, with magneton $\mu_B \equiv e\hbar/2m$:

$$H' = \mu_B B_{\text{ext}}\frac{(\boldsymbol{L} + 2\boldsymbol{S})}{\hbar}\cdot\hat{\boldsymbol{z}}$$

Using perturbation theory we find:

$$E^1 = \mu_B B_{\text{ext}}\frac{\langle\boldsymbol{L} + 2\boldsymbol{S}\rangle}{\hbar}\cdot\hat{\boldsymbol{z}} = \mu_B B_{\text{ext}} g m_j$$

The Landé g-factor is given by:

$$g = 1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}$$

The Stark Effect is splitting from an electric field in atoms like Hydrogen, for example, if $n = 2$, using symmetry:

$$H' = -\boldsymbol{\mu}_e\cdot\boldsymbol{E} = -3e|\boldsymbol{E}|a\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}$$

To first order correction $E_\pm^1 = \pm 3e|\boldsymbol{E}|a$.

Hydrogen Fine Structure

The Hamiltonian for Hydrogen only considers Coulomb attraction and is:

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

A relativistic correction to first order:

$$H'_{\text{rel}} = -\frac{p^4}{8m^3c^2}$$

$$E_{\text{rel}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{\ell + 1/2}\right)$$

Correction due to spin-orbit coupling:

$$H'_{\text{so}} = \frac{e^2}{8\pi\epsilon_0}\frac{1}{m^2c^2r^3}\boldsymbol{S}\cdot\boldsymbol{L}$$

$$E_{\text{so}}^1 = \frac{(E_n)^2}{2mc^2}\left(\frac{2n[j(j+1) - \ell(\ell+1) - 3/4]}{\ell(\ell + 1/2)(\ell + 1)}\right)$$

The fine-structure adjusts by ~ 1 meV:

$$E_{\text{fs}}^1 = E_{\text{rel}}^1 + E_{\text{so}}^1 = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j + 1/2}\right)$$

Dipole-dipole interactions also adjust by ~ 6 μeV in the hyperfine structure.

These energies may also be expressed in terms of the fine-structure constant:

$$\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137.036}$$

These identities help evaluate energies:

$$\langle r^{-1}\rangle = [n^2a]^{-1}$$

$$\langle r^{-2}\rangle = [(\ell + 1/2)n^3a^2]^{-1}$$

$$\langle r^{-3}\rangle = [\ell(\ell + 1/2)(\ell + 1)n^3a^3]^{-1}$$

The Variational Principle

The variational principle gives an upper bound on the ground-state energy:

$$E_{\text{gs}} \leq \langle H\rangle = \langle\psi|H|\psi\rangle$$

Applying to the Harmonic Oscillator:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2;\ \psi = \left(\frac{2b}{\pi}\right)^{1/4}e^{-bx^2}$$

Finding $\langle H\rangle$ and $\langle H\rangle_{\min}$ at $b = m\omega/2\hbar$:

$$\langle H\rangle = \langle T\rangle + \langle V\rangle = \frac{\hbar^2 b}{2m} + \frac{m\omega^2}{8b} = \frac{\hbar\omega}{2}$$

For the Hydrogen atom, $k = e^2/4\pi\epsilon_0$:

$$H = \frac{p^2}{2m} - \frac{k}{r};\ \psi = \left(\frac{2b}{\pi}\right)^{3/4}e^{-br^2}$$

With $\langle H\rangle_{\min}$ at $b = (8/\pi)(2mk/3\hbar^2)^2$:

$$\langle H\rangle = \frac{3\hbar^2 b}{2m} - k\sqrt{\frac{8b}{\pi}} = -\frac{4mk^2}{3\pi\hbar^2} \approx -11.5 \text{ eV}$$

More Applications of V.P.

In the Helium atom, $E_{\text{gs}} = -79.0$ eV, $H = \widetilde{H} + V_{\text{ee}}$, to rough approximation:

$$H \approx \widetilde{H} = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2)$$

Which is quite simply a scaling of E_1:

$$E = -8 \cdot 13.6 \text{ eV} = -108.8 \text{ eV}$$

The coulomb repulsion is given as:

$$V_{\text{ee}} = -\frac{e^2}{4\pi\epsilon_0}\left(\frac{2}{r_1} + \frac{2}{r_2} - \frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\right)$$

Which again may be expressed in E_1:

$$E = (5/2) \cdot 13.6 \text{ eV} = +34.0 \text{ eV}$$

Summing these variational results find:

$$E_1^{\text{He}} = -74.8 \text{ eV}$$

Which is close to the experimental. Closer estimates may also account for shielding of the nucleus be electrons.

For the Hydrogen molecule, we find the Hamiltonian is in terms of momentum and Coulomb forces:

$$H = -\frac{p_1^2 + p_2^2}{2m}! + \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r_{12}} + \frac{1}{R} - \frac{1}{r_1} - \frac{1}{r_1'} - \frac{1}{r_2} - \frac{1}{r_2'}\right)$$

With expressions $D_1, D_2, I, X_1,$ & X_2:

$$\langle H \rangle_{\pm} = 2E_1\left[1 - \frac{a}{R} + \frac{2D_1 - D_2 \pm (2IX_1 - X_2)}{1 \pm I^2}\right]$$

Two-State Statics

Let a two-state system be such that:

$$\begin{Bmatrix} H_0\psi_a = E_a\psi_a \\ H_0\psi_b = E_b\psi_b \end{Bmatrix} \Longleftarrow \langle \psi_a | \psi_b \rangle = \delta_{ab}$$

For wavefunction $\psi = c_a\psi_a + c_b\psi_b$, with unitary evolution under $H\psi = i\hbar\partial_t\psi$:

$$\psi(t) = c_a\psi_a e^{-iE_a t/\hbar} + c_b\psi_b e^{-iE_b t/\hbar}$$

Two State Dynamics

Now, let $H = H_0 + H'(t)$, write the general form of ψ, plug into $H\psi = i\hbar\partial_t\psi$, and use orthogonality to find:

$$\dot{c}_a = -\frac{i}{\hbar}\left(c_a H'_{aa} + c_b H'_{ab} e^{-i(E_b - E_a)t/\hbar}\right)$$

$$\dot{c}_b = -\frac{i}{\hbar}\left(c_b H'_{bb} + c_a H'_{ba} e^{-i(E_a - E_b)t/\hbar}\right)$$

For small perturbations with $H'_{aa} = 0$, $H'_{bb} = 0$, $\omega_0 = (E_b - E_a)/\hbar$, and $c_a(0) = 1$, $c_b(0) = 0$, first order is:

$$c_a^{(1)}(t) = 1$$

$$c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t d\bar{t}\; H'_{ba}(\bar{t}) e^{i\omega_0\bar{t}}$$

Sinusoidal Perturbations

For sinusoidal time dependence to the perturbation, assumptions as before, with $\omega \sim \omega_0$, transition probability is $P_{a\to b}(t) = |c_b(t)|^2$, or in these limits:

$$P_{a\to b}(t) \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}$$

Light has just such a time dependence, which leads to absorption and stimulated emission with probability above.

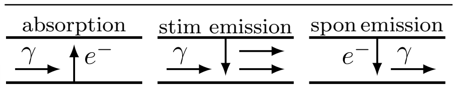
With incoherent light from all directions, we take the integral over ω of spectral density $\rho(\omega)$ and directions:

$$P_{b\to a} = \frac{\pi |q\, r_{ba}|^2}{3\epsilon_0\hbar^2}\rho(\omega_0)\; t \quad \propto \quad t$$

Spontaneous emission occurs at:

$$P_{b\to a} = \frac{\omega_0^3 |q\, r_{ba}|^2}{3\pi\epsilon_0\hbar c^3}\; t \quad \propto \quad t$$

Emission is governed by selection rules. For hydrogenic atoms find: $\ell' - \ell = \pm 1$, $m' - m = 0, \pm 1$.



If we rather consider transition from a discrete state, a, to a continuum, b, we find for DOS, D, Fermi's Golden Rule:

$$P_{a\to b} = \frac{\pi |V_{ab}|^2}{2\hbar} D(E_b)\; t \quad \propto \quad t$$

Rabi Oscillations

With $V_{ab} = V_{ba}$, and the perturbation:

$$H' = \begin{pmatrix} H'_{aa} H'_{ab} \\ H'_{ba} H'_{bb} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & V_{ab}e^{i\omega t} \\ V_{ba}e^{-i\omega t} & 0 \end{pmatrix}$$

Through direct substitution:

$$\dot{c}_a = -\frac{i}{\hbar}\left(c_b V_{ab} e^{+i\omega t} e^{-i(E_b - E_a)t/\hbar}\right)$$

$$\dot{c}_b = -\frac{i}{\hbar}\left(c_a V_{ba} e^{-i\omega t} e^{-i(E_a - E_b)t/\hbar}\right)$$

With differentiation and substitution, and $\omega_r = \sqrt{(\omega_0 - \omega)^2 + |V_{ab}|^2/\hbar^2}/2$, find:

$$P_{a\to b} = \left(\frac{V_{ba}}{2\hbar\omega_r}\right)^2 \sin^2(\omega_r t)$$

This becomes the perturbation result if $\omega_r \to (\omega_0 - \omega)/2$, or $|V_{ab}|^2 \ll \hbar^2(\omega - \omega_r)^2$

Note that the prefactor is a Lorentzian of width $\Delta w = |V_{ab}|/\hbar$, and if $\omega = \omega_0$, transition are $P_{a\to b}(t) = \sin^2(V_{ab}t/2\hbar)$.

Magnetic Resonance

For magnetic field $B_0\,\hat{\boldsymbol{z}}$, and transverse field, $B_{\text{rf}}(\cos(\omega t)\hat{\boldsymbol{x}} - \sin(\omega t)\hat{\boldsymbol{y}})$, find:

$$H = -\gamma \boldsymbol{B}\cdot\boldsymbol{S} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_{\text{rf}}e^{i\omega t} \\ B_{\text{rf}}e^{-i\omega t} & -B_0 \end{pmatrix}$$

For $\chi = \binom{a}{b}$, $H\chi = i\hbar\partial_t\chi$, and $\Omega = \gamma B_{\text{rf}}$:

$$\dot{a} = \frac{i}{2}\left(\Omega e^{+i\omega t} b + \omega_0 a\right)$$

$$\dot{b} = \frac{i}{2}\left(\Omega e^{-i\omega t} a - \omega_0 b\right)$$

Probability of a spin-flip starting from $|a\rangle$ is, with $\omega' \equiv \sqrt{(\omega - \omega_0)^2 + \Omega^2}$:

$$P_b(t) = \left|\frac{\Omega}{\omega'}\sin(\omega' t/2)\right|^2 = \frac{\Omega^2 \sin^2(\omega' t/2)}{(\omega - \omega_0)^2 + \Omega^2}$$

Plotting the prefactor is a Lorentzian centered at ω_0 and width $\Delta w = 2\Omega$.

Scattering

We assume the scattering wavefunction is an incoming plane wave and an outgoing spherical wave for large r:

$$\psi(r, \theta) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

Expand the scattering amplitude as:

$$f(\theta) = \sum_{\ell=0}^{\infty} a_\ell (2\ell + 1) P_\ell(\cos(\theta))$$

Differential cross section $D(\theta) = |f(\theta)|^2$. The total scattering cross section is:

$$\sigma = \int d\Omega\; D(\theta) = 4\pi \sum_{\ell=0}^{\infty}(2\ell + 1)|a_\ell|^2$$

In one-dimensional systems, scattering may be formulated as a phase shift δ:

$$\psi(x) = A\left[e^{ikx} - e^{i(2\delta - kx)}\right]$$

Here, the scattering amplitude is:

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell + 1)e^{i\delta_\ell}\sin(\delta_\ell) P_\ell(\cos(\theta))$$

Likewise, the scattering cross section is:

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell + 1)\sin^2(\delta_\ell)$$

Identical Scattering

For identical particles scattering, find:

$$\psi(r) = e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}} + e^{-i\boldsymbol{k}_0\cdot\boldsymbol{r}} + [f(\theta) + f(\pi - \theta)]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{|\boldsymbol{r}|}$$

Bosons have only 1 singlet; fermions have 3 triplets too ($|f(\theta) - f(\pi - \theta)|^2$):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{bosons}} = \frac{4}{4}\cdot|f(\theta) + f(\pi - \theta)|^2$$

First Born Approximation

Schrödinger Equation, integral form:

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) - \frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}_0 \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}_0|}}{|\boldsymbol{r}-\boldsymbol{r}_0|} V(\boldsymbol{r}_0)\psi(\boldsymbol{r}_0)$$

Assume that the scattering potential is local to a region $\boldsymbol{r}_0$, and that we look from far away, with $\psi(\boldsymbol{r}_0) \approx \psi_0(\boldsymbol{r}_0)$, i.e. the limit of a weak scattering field:

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2 A}\int d^3\boldsymbol{r}_0\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_0} V(\boldsymbol{r}_0)\psi(\boldsymbol{r}_0)$$
$$= -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}_0\ e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}_0} V(\boldsymbol{r}_0)$$

For low energy scattering, we find:

$$f(\theta,\varphi) \approx -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}_0\ V(\boldsymbol{r}_0)$$

For a spherically symmetric potential, with $\kappa \equiv 2k\sin(\theta/2)$, φ independence:

$$f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0^\infty dr\ r\,V(r)\sin(\kappa r)$$

Entanglement

Entanglement: the wavefunction isn't factorizable into single-particle states.

The π^0 particle has zero charge and spin. In the decay $\pi^0 \to e^- + e^+$, the e^- and e^+ must be in the singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{e^-}|\downarrow\rangle_{e^+} - |\downarrow\rangle_{e^-}|\uparrow\rangle_{e^+}\right)$$

Measurement of the e^- and e^+ must always yield one in $|\uparrow\rangle$ and one in $|\downarrow\rangle$ — the measurement determines which, even if the particles are far apart.

Bell's Theorem: local hidden variables can't predict experimental behavior. For example, measuring singlet decay with two detectors at angles traces a cosine on average, not a triangle wave.

Adiabatic Theorem

In a gapped spectrum, with slow (adiabatic) transformations to H that retain gaps, nth state remains nth state. For dynamic and geometric phases Θ and γ, the state undergoes unitary evolution:

$$\Theta_n(t) \equiv -\frac{1}{\hbar}\int_0^t d\bar{t}\ E_n(\bar{t})$$
$$\gamma_n(t) \equiv i\int_0^t d\bar{t}\ \langle\psi_n|\partial_t\psi_n\rangle$$
$$= i\int_{R_i}^{R_f} dR\ \langle\psi_n|\partial_R\psi_n\rangle$$

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