One Sentence Summary

Upon measurement, the wave function of a particle assumes an eigenfunction of that measurement operator, and not all measurement operators have the same eigenfunctions.

Chapter 1

Solution of the Schrödinger Equation determines the future system behavior:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$$

The probability of a particle in (a,b):

$$P_{(a,b)} = \int_a^b dx\ \|\Psi(x,t)\|^2$$

The probability of a particle anywhere:

$$P_{(\text{space})} = \int_x dx\ \|\Psi(x,t)\|^2 = 1$$

Not all wave functions correspond to a particle ($\Psi(x) = 0$ or $\lim_{x\to\infty}\Psi(x) \neq 0$), otherwise, they may be normalized once so $P_{(\text{space})} = 1$ holds for all time.

The expectation value of $f(x)$:

$$\langle f(x)\rangle = \int_x dx\ \Psi^*(x,t)f(x)\Psi(x,t)$$

The standard deviation of $f(x)$:

$$\sigma(f(x)) = \sqrt{\langle (f(x))^2\rangle - \langle f(x)\rangle^2}$$

The expectation value of position:

$$\langle x\rangle = \int_x dx\ \Psi^*(x,t)\,[x]\,\Psi(x,t)$$

The expectation value of momentum:

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \int_x dx\ \Psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\Psi$$

These are useful because: "all classical dynamical variables can be expressed in terms of position and momentum."

The de Broglie formula relates:

$$p = \frac{2\pi\hbar}{\lambda}$$

Now: "the more precisely determined a particle's position is, the less precisely determined a particle's momentum is." Which is the Uncertainty Principle:

$$\sigma(x)\sigma(p) \geq \frac{\hbar}{2}$$

Chapter 2

If $V(x,t)$ is independent of t then:

$$\Psi(x,t) = \psi(x)\varphi(t)$$

Which may be written as two ODEs, with solving and $H = p^2/2m + V(x)$:

$$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi \implies \varphi(t) = \exp\left(-\frac{iEt}{\hbar}\right)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \implies H\psi = E\psi$$

Properties of separable solutions, $\psi(x)$:

- the probability density is time independent for "stationary states"
- wave function is a linear combination of the separable solutions
- definite total energy for solutions

Assuming completeness of $\{\psi_n\}$:

$$\Psi(x,t) = \sum_n c_n\psi_n \exp\left(-\frac{iEt}{\hbar}\right)$$

Where the projection operator finds:

$$c_n = \int_x dx\ \psi_n^* f(x)$$

The probability of an eigenfunction, and probability of all eigenfunctions:

$$P(E_n) = |c_n|^2 \quad \text{and} \quad \sum_n |c_n|^2 = 1$$

Hamiltonian's eigenvalues are energies:

$$\langle H\rangle = \int_x dx\ \Psi^*[H]\Psi = E\int_x dx\ \|\Psi\|^2 = E$$

Hamiltonian's expectation value is continuous, but the energies are discrete:

$$\langle H\rangle = \sum_n |c_n|^2 E_n \quad \text{and} \quad \sigma(H) = 0$$

Infinite Square Well

Potential for the infinite square well:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x \leq a \\ \infty & a < x \end{cases}$$

The SEQ is then, with $k = \sqrt{2mE/\hbar^2}$:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\psi \implies \frac{d^2\Psi}{dx^2} = -k^2\Psi$$

The general solution to this ODE is:

$$\psi(x) = A\sin(k_n x) + B\cos(k_n x)$$

Normalizing and considering boundary values, $A = \sqrt{2/a}$ and $k_n = n\pi/a$:

$$\Psi(x) = \begin{cases} 0 & x < 0 \\ A\sin(k_n x) & 0 \leq x \leq a \\ 0 & a < x \end{cases}$$

Infinite Square Well (continued)

Note, k is the free space wave vector:

$$\hbar k = \hbar\sqrt{\frac{2m}{\hbar^2}\cdot E} = \sqrt{2m\cdot\frac{p^2}{2m}} = p$$

Solving the permitted momenta for E:

$$E_n = T_n = \frac{p_n^2}{2m} = \frac{\hbar^2}{2m}\cdot k_n^2 = \frac{\pi^2\hbar^2}{2ma^2}\cdot n^2$$

The stationary states are orthonormal:

$$\int_x dx\ \psi_m^*\psi_n = \delta_{m,n}$$

Where the Kronecker Delta is defined:

$$\delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Algebraic Harmonic Oscillator Solution

Potential for the harmonic oscillator:

$$V(x) = \frac{m\omega^2}{2}x^2$$

The SEQ for the harmonic oscillator is:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \frac{m\omega^2x^2}{2}\Psi = E\Psi$$

Define "raising," "lowering" operators:

$$a_\pm = (\mp ip + m\omega x)/\sqrt{2\hbar m\omega}$$

Position and momentum operators are:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-);\ p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

With substitution, canonical commutation $[x,p] = i\hbar$, and $[a_-,a_+] = 1$:

$$H = \hbar\omega(a_+a_- + 1/2)$$

The operator eigenvalue problem is:

$$H(a_\pm\psi_n) = (E\pm\hbar\omega)a_\pm\psi_n;\ E_n = (2n+1)\hbar\omega/2$$

Assuming there is a minimum state where $a_-\psi_0 = 0$, and solving the ODE:

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right);\ E_0 = \frac{\hbar\omega}{2}$$

Raising and lowering eigenfunctions while maintaining normalization:

$$\frac{a_+}{\sqrt{n+1}}\cdot\psi_n = \psi_{n+1};\quad \frac{a_+}{\sqrt{n}}\cdot\psi_n = \psi_{n-1}$$

The normalized n-th eigenfunction is:

$$\psi_n = \frac{(a_+)^n}{\sqrt{n!}}\psi_0$$

The eigenfunctions of the harmonic oscillator are orthonormal:

$$\int_x dx\ \psi_m^*\psi_n = \delta_{m,n}$$

Analytic Harmonic Oscillator Solution

The SEQ for the harmonic oscillator is:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+\frac{m\omega^2x^2}{2}\Psi=E\Psi$$

With $\xi=\sqrt{m\omega/\hbar}x$, and $K=2E/\hbar\omega$:

$$\frac{d^2\Psi}{d\xi^2}=(\xi^2-K)\Psi$$

For ξ large $\Psi=A\exp(-\xi^2/2)$, guess:

$$\Psi(\xi)=h(\xi)\exp(-\xi^2/2)$$

Now, to solve the ODE analytically:

- Plug in the ansatz to the ODE
- Algebraically simplify if possible
- Guess power series solution for h
- Plug in the power series to ODE
- Unify the exponent of the powers
- Equate to zero/eliminate powers
- Solve for the recursion relation

The recursion relation is:

$$a_{j+2}=\frac{2j+1-K}{(j+1)(j+2)}a_j$$

These solutions are normalizable iff:

$$K=2n+1 \implies E_n=(n+1/2)\hbar\omega$$

With constants a_0 and a_1 to normalize:

$$\psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{H_n(\xi)}{\sqrt{2^n n!}}\exp(-\xi^2/2)$$

The first few Hermite Polynomials are:

$H_0=1$ $\quad$ $H_1=2\xi$

$H_2=4\xi^2-2$ $\quad$ $H_3=8\xi^3-12\xi$

$H_4=16\xi^4-48\xi^2+12$ $\quad$ $H_5=32\xi^5-160\xi^3+120\xi$

Free Particle

For potential $V(x)=0$, $k=\sqrt{2mE}/\hbar$:

$$\Psi(x,0)=A\exp(ikx)+B\exp(-ikx)$$

Let $k<0$ be left, and $0<k$ be right:

$$\Psi_k(x,t)=A\exp(i(kx-\hbar k^2t/2m))$$

Which, using a linear superposition is:

$$\Psi(x,t)=\sum_k\Psi_k(x,t)$$

Using the De Broglie Hypothesis:

$$p=h/\lambda \text{ and } k=2\pi/\lambda \implies p=\hbar k$$

If given $\Psi(x,0)$, then to find $\Psi(x,t)$:

$$\Phi(k)=\frac{1}{\sqrt{2\pi}}\int_x dx\ \Psi(x,0)e^{-ikx}$$

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_k dk\ \Phi(k)e^{i(kx-\hbar k^2t/2m)}$$

Plancherel's theorem is:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_k dk\ F(k)e^{ikx}\iff F(k)=\frac{1}{\sqrt{2\pi}}\int_x dx\ f(x)e^{-ikx}$$

For any wave packet, ex. $\omega=\hbar k^2/2m$:

$$v_{\text{phase}}=\frac{\omega}{k};\qquad v_{\text{group}}=\frac{d\omega}{dk}$$

Delta Function Potential

For $E<V(\pm\infty)$ a state is bound to a region, and for $E>V(\pm\infty)$ a state is not bound. Letting $V(\pm\infty)=0$, then:

$$\begin{cases}\text{bound: } E<0\Rightarrow\Psi=\sum_n\ldots\ \Rightarrow \text{normalizable}\\ \text{scatter: } E>0\Rightarrow\Psi=\int_k dk\ \ldots\Rightarrow\text{unnormalizable}\end{cases}$$

The "delta function" is defined so that:

$$\int_x dx\ \delta(x-a)f(x)=f(a)$$

Integrating the Schrödinger Equation:

$$\Delta\Psi'=-2m\alpha\psi(0)/\hbar^2$$

If $V(x)=-\alpha\delta(x)$, and $E<0$ then:

- Write Schrödinger Equation
- Use an exponential ansatz
- Use the continuity of Ψ
- Use (dis)-continuity of Ψ'
- Normalization for Ψ
- Solve for E from ansatz

$$E=-\frac{m\alpha^2}{2\hbar^2};\qquad\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$$

If $V(x)=-\alpha\delta(x)$, and $E>0$ then:

- Follow steps as above …
- Normalization for $\Psi \implies$ ☹

$$\psi_-=Ae^{ikx}+Be^{-ikx};\ \psi_+=Fe^{ikx}+Ge^{-ikx}$$

The "normalization" coefficients are given by the experimental set up, and not by any constraints of the potential.

- Does wave originate left or right?
- What amplitude of incident?
- What amplitude of reflected?
- What amplitude of transmitted?

From conditions on Ψ and Ψ', one finds reflection and transmission coefficients:

$$R=\|\text{reflected}\|^2/\|\text{in}\|^2=(1+(2\hbar^2E/m\alpha^2))^{-1}$$

$$T=\|\text{transmit}\|^2/\|\text{in}\|^2=(1+(m\alpha^2/2\hbar^2E))^{-1}$$

Where, with conservation, $R+T=1$.
Tunneling is transmission if $E<V_{\text{max}}$.

Finite Square Well

$V(x)=-V_0$ for $-a<x<a$, else $V(x)=0$.
Now, consider the bound state where $-V_0<E<0$, and $l\equiv\sqrt{2m(E+V_0)/\hbar^2}$:

$$\psi(x)=\begin{cases}F\exp(+kx) & x<-a\\ D\cos\text{ or }\sin(lx) & -a<x<a\\ F\exp(-kx) & a<x\end{cases}$$

Use Ψ and Ψ' to find k, graphically via transcendental equation, and thence E.
Within limits, the solutions behave as:

- Wide deep: infinite square well truncated and translated by $-V_0$
- Wide shallow: free particle
- Narrow deep: delta function
- Narrow shallow: one bound state

Finite Square Well (cont)

For scattering states, $E>0$, and:

$$\frac{1}{T}=\frac{1}{1-\sigma_{\text{tot}}}=1+\frac{V_0^2}{4E(E+V_0)}\sin^2(2al)$$

At some energies the transmission is 1. This is the Ramsauer-Townsend effect.

Hilbert Space

Everything you can know for a system is stored in a "state ket," or, vector $|v\rangle$. Operators, or "observables" are "linear transformations," or as matrices that extract information about the system.

The "Hilbert Space" is the vector space of "square-integrable" wave functions.

The inner product of two functions is:

$$\langle f|g\rangle\equiv\int_a^b dx\ f^*(x)g(x)$$

If f and g live in Hilbert Space, then the inner product will be bounded by the Schwartz ineq $\langle f|g\rangle\le\sqrt{\langle f|f\rangle\langle g|g\rangle}$

- Normalized: $\langle f|f\rangle=1$
- Orthogonal: $\langle f|g\rangle=0$
- Orthonormal: $\langle f|g\rangle=0$, $\langle f|f\rangle=1$
- Complete: $g(x)=\sum_n c_nf_n(x)$
- Complete orthon.: $c_n=\langle f_n|f\rangle$

Inner product, or "bracket" of $|\alpha\rangle$, $|\beta\rangle$:

$$\langle\alpha|\beta\rangle=a_1^*b_1+a_2^*b_2+\ldots a_n^*b_n$$

Observables

Note that operators act on kets from the left and act on bras from the right.

Measurements observe real values and thence expect real expecation values-observables are Hermitian Operators:

$$\langle Q\rangle=\langle\Psi|Q\Psi\rangle=\langle\Psi Q|\Psi\rangle=\langle\Psi|Q|\Psi\rangle$$

Hermitian Conjugate, or adjoint is $Q^\dagger$:

$$\langle f|Qg\rangle=\langle fQ^\dagger|g\rangle$$

Determinate states are the values that you keep measuring if you repeatedly measure observable system properties. This determinacy means $\sigma(Q)=0$, or:

$$\sigma(Q)=\sqrt{\langle(\boldsymbol{Q}-\langle Q\rangle\boldsymbol{1})^2\rangle}$$

$$=\sqrt{\langle\Psi\,|(\boldsymbol{Q}-q\boldsymbol{1})^2|\,\Psi\rangle}$$

$$\implies\boldsymbol{Q}-q\boldsymbol{1}=\boldsymbol{0}$$

Which is precisely the condition for q to be an eigenvalue of the operator Q.

- "The collection of all eigenvalues of an operator is its spectrum."
- If two kets share an eigenvalue, that eigenvalue and the spectrum are said to be "degenerate."

Hermitian Operator Eigenfunctions

Generalizing δ-function observations:

discrete $\iff$ normalizable $\iff$ realizable
continuous $\Leftrightarrow$ un-normalizable $\Leftrightarrow$ un-realizable

Hermitian operator eigenvalues are real, and their eigenvectors are orthogonal, and in addition, they are a complete set.

Statistical Interpretation

Measurements of a particle in state Ψ by observable Q yields an eigenvalue q_n, where the probability of observing q_n is $|c_n|^2$ for discrete $c_n = \langle f_n|\Psi\rangle$, or $|c(x)|^2 dx$ for continuous $c(x) = \langle f_x|\Psi\rangle$.

Uncertainty Principle

The generalized uncertainty principle for non-commuting observables A, B:

$$\sigma(A)^2\sigma(B)^2 \geq \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2$$

This is found by writing the standard deviations, comparing the imaginary part of the Schwarz Inequality, expanding, and writing as a commutator.

"Measurement collapses the wave function to a narrow spike, which necessarily carries a broad range of momenta"

The minimum-uncertainty wave packet: $\Psi(x) = A\exp(-a(x-\langle x\rangle)^2)\exp(i\langle p\rangle x/\hbar)$
Using the Schrödinger Eq., it is found:

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[A,B]\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle$$

Substituting H, Q in the generalized uncertainty relation, and then letting $\Delta E \equiv \sigma(H)$, with $\Delta t \equiv \sigma(Q)/|d\langle Q\rangle/dt|$:

$$\sigma(H)\sigma(Q) \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right| \implies \Delta E\Delta t \geq \frac{\hbar}{2}$$

Vectors and Operators

Operators act on vectors as $A|\alpha\rangle = |\beta\rangle$.
Coefficients are given as $c_n = \langle n|\Psi(t)\rangle$.
In general vectors are $|\Psi(t)\rangle = \sum c_n|e_n\rangle$.
Matrix elements are $\langle e_m|A|e_n\rangle = A_{mn}$.

Completeness: $\sum|e_n\rangle\langle e_n| = \mathbf{1} = \int|e_z\rangle\langle e_z|$.

$$\mathbf{1} = \sum|n\rangle\langle n| = \int dp\,|p\rangle\langle p| = \int dx\,|x\rangle\langle x|$$

Consider the two state system:

$$H = \begin{pmatrix} h & g \\ g & h\end{pmatrix} \Rightarrow \begin{vmatrix} h-E & g \\ g & h-E\end{vmatrix} = 0 \Rightarrow E_\pm$$

$$\begin{pmatrix} h-E_\pm & g \\ g & h-E_\pm\end{pmatrix}\begin{pmatrix} a_\pm \\ b_\pm\end{pmatrix} = \begin{pmatrix} 0 \\ 0\end{pmatrix} \Rightarrow \text{normalize}$$

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