Maxwell's Equations

For the scalar and vector potentials:
$E = -\nabla V - (1/c)\partial_t \boldsymbol{A}; \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$

Now, Maxwell's equations in SI units (left) and Gaussian CGS units (right).

Charges are the source of electric fields:
$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \iff \nabla \cdot \boldsymbol{E} = 4\pi\rho$$
There is no magnetic charge:
$$\nabla \cdot \boldsymbol{B} = 0 \iff \nabla \cdot \boldsymbol{B} = 0$$
A changing magnetic field induces $\boldsymbol{E}$:
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \iff \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$
$\boldsymbol{B}$ originates in currents, induction too:
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \Leftrightarrow \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

Electrostatics in SI

Lorentz force law for charged particles:
$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$
For a point particle at the origin:
$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}; \ \boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}; \ \boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$$
Poisson's equation for scalar potential:
$$\nabla^2 V = -\rho/\epsilon_0$$
The scalar potential by integration:
$$V(x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(x')}{|x - x'|}$$
Poisson's equation for vector potential:
$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$
The vector potential by integration:
$$\boldsymbol{A}(x) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\boldsymbol{J}(x')}{|x - x'|}$$

Electrostatics in CGS

Lorentz force law for charged particles:
$$\boldsymbol{F} = q(\boldsymbol{E} + (\boldsymbol{v}/c) \times \boldsymbol{B})$$
For a point particle at the origin:
$$V = \frac{q}{r}; \qquad \boldsymbol{E} = \frac{q}{r^2}\hat{\boldsymbol{r}}; \qquad \boldsymbol{F} = \frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$$
Poisson's equation for scalar potential:
$$\nabla^2 V = -4\pi\rho$$
The scalar potential by integration:
$$V(x) = \int d^3x' \frac{\rho(x')}{|x - x'|}$$
Poisson's equation for vector potential:
$$\nabla^2 \boldsymbol{A} = -(4\pi/c)\boldsymbol{J}$$
The vector potential by integration:
$$\boldsymbol{A}(x) = \frac{1}{c}\int d^3x' \frac{\boldsymbol{J}(x')}{|x - x'|}$$

Electrodynamics

Current density is, noting $\boldsymbol{I} = \int \boldsymbol{J} \cdot d\boldsymbol{a}$:
$$\boldsymbol{J} = nq\boldsymbol{v}_D = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \approx \sigma \boldsymbol{E}$$
This is equivalent to Ohm's law:
$$\boldsymbol{J} = \sigma \boldsymbol{E} \iff \mathcal{E} = IR$$
The Joule heating law relates power:
$$P = I\mathcal{E} = I^2 R = \mathcal{E}^2/R$$
The electromotive force is given by:
$$\mathcal{E} = \oint \boldsymbol{E} \cdot d\boldsymbol{l}$$
Magnetic flux is defined as:
$$\Phi \equiv \int \boldsymbol{B} \cdot d\boldsymbol{a}$$
The flux rule means that a changing magnetic field induces an electric field:
$$\mathcal{E} = -\frac{d\Phi}{dt} \iff \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
Lenz's law: induced current opposes a change in flux, to minimize flux change.

Induction

Inductance between two loops a and b:
$$\Phi_b = \int \boldsymbol{B}_a \cdot d\boldsymbol{a}_b \Leftrightarrow \Phi_a = \int \boldsymbol{B}_b \cdot d\boldsymbol{a}_a$$
Flux relates by the mutual inductance, where in the electron's frame there is an electric field even though in the rest frame there is only a magnetic field:
$$\Phi_b = M_{ba} I_a \Leftrightarrow \Phi_a = M_{ab} I_b$$
Mutual inductances are related as:
$$M_{ab} = M_{ba}$$
The self inductance of a current loop:
$$\Phi = LI \iff \mathcal{E} = -L\,\partial_t I$$

Work

Note the general form of conservation:
$\partial_t(\text{density of stuff}) + \nabla \cdot (\text{current of stuff}) = 0$

Energy contained in the electric field:
$$W = \frac{\epsilon_0}{2}\int_{\text{space}} E^2 d\tau$$
Energy contained in the magnetic field:
$$W = \frac{1}{2\mu_0}\int_{\text{space}} B^2 d\tau$$
The energy density in SI and Gaussian, (in CGS $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units):
$$w = \frac{\epsilon_0}{2}|\boldsymbol{E}|^2 + \frac{1}{2\mu_0}|\boldsymbol{B}|^2 \Leftrightarrow \frac{1}{8\pi}(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)$$
Energy in a capacitor and inductor:
$$W_{\text{capacitor}} = \frac{1}{2}CV^2; \quad W_{\text{inductor}} = \frac{1}{2}LI^2$$

Circuits

Ohm's law and the flux rule can help:
$$\mathcal{E} = IR; \qquad \mathcal{E} = -\partial_t \Phi; \qquad I \equiv \partial_t Q$$
Batteries, Capacitors, and Inductors:
$$\mathcal{E} = \mathcal{E}_0; \quad \mathcal{E} = \frac{Q}{C}; \quad \mathcal{E} = -L\frac{dI}{dt} = -L\frac{d^2Q}{dt^2}$$
Note that for mutual inductance:
$$\mathcal{E} = -\frac{d\Phi}{dt} = -M_{ba}\frac{dI_a}{dt} \text{ (or) } - M_{ab}\frac{dI_b}{dt}$$
Solutions to these ODEs are sines and cosines or exponentials. Consider $t = 0$.

Electromagnetic Waves

The wave equation is in 1D and 3D:
$\partial_{xx} f = (1/v^2)\partial_{tt} f \iff \nabla^2 f = (1/v^2)\partial_{tt} f$

General solution to the wave equation is a superposition of right and left:
$$f(x,t) = r(x - vt) + l(x + vt)$$
Sinusoidal waves are a good sample, (amplitude A, wave number k, phase ϕ):
$$f(x,t) = A\cos[k(x - vt) + \phi]$$
Period, frequency, angular frequency:
$$T = \frac{2\pi}{kv}; \quad \nu = \frac{1}{T}; \quad \omega = 2\pi\nu$$
Waves may be expressed as complex exponentials, if they aren't multiplied.

Electromagnetic induction means that electromagnetic waves can propagate in the absence of currents or charges:
$$\nabla^2 \boldsymbol{E} = \mu_0\epsilon_0\,\partial_{tt}\boldsymbol{E}; \quad \nabla^2 \boldsymbol{B} = \mu_0\epsilon_0\,\partial_{tt}\boldsymbol{B}$$
Electromagnetic waves propagate with $v = c \equiv 1/\sqrt{\mu_0\epsilon_0}$, and can be framed as:
$$\boldsymbol{E} \parallel \hat{\boldsymbol{z}}; \qquad \boldsymbol{B} \parallel \hat{\boldsymbol{y}}; \qquad \boldsymbol{v} \parallel \boldsymbol{E} \times \boldsymbol{B}$$
A simple form with $|\boldsymbol{E}| = |\boldsymbol{B}|$ in CGS:
$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} = E_0\cos(\omega t - \phi)\hat{\boldsymbol{E}}$$
$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} = B_0\cos(\omega t - \phi)\hat{\boldsymbol{B}}$$
Where the directions are defined as:
$$\hat{\boldsymbol{k}} = \text{propagation}$$
$$\hat{\boldsymbol{n}} = \text{polarization}$$
$$\hat{\boldsymbol{E}} = \hat{\boldsymbol{n}}$$
$$\hat{\boldsymbol{B}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}$$

Retarded Time Potentials

Retarded time accounts for the speed of light in solving Maxwell's Equations where the integrals are as before, with:
$$t_r = t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}$$

Poynting Vector, Power

Now, converting to Gaussian units, the Poynting vector, or energy flux is given:

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}$$

The total power radiated through a volume is then (often time-dependent):

$$P = \int_{\text{volume}} r^2 \sin(\theta) dr\, d\theta\, d\varphi\, \boldsymbol{S}$$

It may be useful to recall that:

$$\int_0^{\pi} d\theta\ \sin^3(\theta) = \frac{4}{3}$$

Dipole Radiation

In general, the dipole moment is:

$$\boldsymbol{P} = \int_V d^3r'\ \rho(\boldsymbol{r'})\boldsymbol{r'}$$

For a dipole moving with period ω, in the far field approximation $|\boldsymbol{x}| \gg |\boldsymbol{x'}|$, with $k = \omega/c$ and $r = |\boldsymbol{x}|$:

$$\boldsymbol{A} = \frac{1}{c}\int d^3x' \frac{\boldsymbol{J}(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} e^{ik|\boldsymbol{x}-\boldsymbol{x'}|} e^{i\omega t}$$

$$\approx \frac{1}{c}\frac{e^{-ikr}}{r} e^{i\omega t}\int d^3x' \boldsymbol{J}(\boldsymbol{x'})$$

From this, the dipole fields are then:

$$\boldsymbol{B} = k^2\frac{e^{-ikr}}{r}\boldsymbol{n}\times\boldsymbol{P}; \qquad \boldsymbol{E} = -\boldsymbol{n}\times\boldsymbol{B}$$

The Poynting vector then becomes:

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}|\boldsymbol{B}|^2\hat{\boldsymbol{n}}$$

Time-average of the Poynting vector is:

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi}\frac{k^4}{r^2}|\boldsymbol{n}\times\boldsymbol{p}|^2\hat{\boldsymbol{n}}$$

It may be useful to recall that:

$$|\boldsymbol{A}\times\boldsymbol{B}|^2 = [|\boldsymbol{A}||\boldsymbol{B}|\sin(\theta)]^2$$

The Larmor formula for power is then:

$$P = \frac{2}{3}\frac{\ddot{p}^2}{c^3}$$

Moving Point Charges

The Liénard-Wiechert potentials for moving charges are given in CGS as:

$$V(\boldsymbol{r},t) = \left[\frac{qc}{|\boldsymbol{r}-\boldsymbol{r'}|c - (\boldsymbol{r}-\boldsymbol{r'})\cdot\boldsymbol{v}_q}\right]_{\text{ret}}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \left[\frac{v}{c^2}V(\boldsymbol{r},t)\right]_{\text{ret}}$$

From these, with some effort, one may derive the Larmor formula for power:

$$P = \frac{2}{3}\frac{q^2}{c^3}\dot{v}^2$$

Special Relativity

Special relativity postulates that you can't tell one inertial frame from another, and the speed of light is a universal constant independent of the relative motion of the source and receiver.

For velocity v between frames, we let:

$$\beta = \frac{v}{c}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

It happens to be convenient to express quantities as 4-vectors, $A = (A_0, \boldsymbol{A})$.

Lorentz Transform between frames is:

$$L = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4-vectors transform as $A' = LA$. Note that the metric $A\cdot B = A_0B_0 - \boldsymbol{A}\cdot\boldsymbol{B}$ is invariant under Lorentz Transform.

The "light cone" dictates causality:

$$s^2 = c^2(t_2^2 - t_1^2) - |\boldsymbol{x}_2 - \boldsymbol{x}_1|^2$$

$s^2 > 0$ timelike

$s^2 < 0$ spacelike

$s^2 = 0$ lightlike

The proper time τ is the time in the particle's rest frame, and generally:

$$d\tau = \frac{dt}{\gamma(t)}$$

With moving sources/recievers, wavelenth may red or blue shift (Doppler):

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

Useful relations from conservation:

$$p = \gamma m v; \quad E = \gamma m c^2$$

$$E = \sqrt{m^2c^4 + c^2p^2}$$

In the ultra-relativistic limit:

$$p = \gamma m v; \qquad E = c|p|$$

In the non-relativistic limit:

$$p = mv; \qquad E = mc^2 + p^2/2m$$

Field-strength tensor $F = \partial^\alpha A^\beta - \partial^\beta A^\alpha$:

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Tensors transform as $F' = LFL^T$, which means that a change of frames changes the electromagnetic fields.

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