Maxwell's Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla\times \boldsymbol{E} = 0 - \frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla\times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Elementary Results

Lorentz Force on a charged particle:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$

Gauss's law, from Green's theorem:

$$\oint \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0}\int_V \rho d\tau = \frac{q_{\text{enc}}}{\varepsilon_0}$$

Point charge: $\boldsymbol{E} = q/(4\pi\varepsilon_0\ r^2)\hat{\boldsymbol{r}}$, and from $\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0$, $\nabla\cdot(\hat{\boldsymbol{r}}/r^2) = 4\pi\delta(\boldsymbol{r})$
Infinite line: $\boldsymbol{E} = \lambda/(2\pi\varepsilon_0 s)\hat{\boldsymbol{s}}$
Infinite sheet: $\boldsymbol{E} = \sigma/(2\varepsilon_0)\hat{\boldsymbol{n}}$
Cylinder in: $\boldsymbol{E} = \rho r/(2\varepsilon_0)\hat{\boldsymbol{s}}$
Cylinder out: $\boldsymbol{E} = \rho a^2/(2\varepsilon_0 r)\hat{\boldsymbol{s}}$

Coulomb's law for two point particles:

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}$$

Vector Calculus

Vector operators are:

- ∇ Gradient is the direction
- $\nabla\cdot$ Divergence is the source
- $\nabla\times$ Curl is the amount of swirl

The gradient of a scalar function is:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

The divergence of a vector function is:

$$\nabla\cdot\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

The curl of a vector function is:

$$\nabla\times\boldsymbol{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Green's Theorem:

$$\int_V (\boldsymbol{\nabla}\cdot\boldsymbol{v})d\tau = \oint_S \boldsymbol{v}\cdot d\boldsymbol{a}$$

Stoke's Theorem:

$$\int_S (\boldsymbol{\nabla}\times\boldsymbol{v})d\boldsymbol{a} = \oint_l \boldsymbol{v}\cdot d\boldsymbol{l}$$

- dl sum of three inf. translations
- da product of two inf. translations in the direction of the third
- $d\tau$ product of three inf. translations

Superposition Principle

"Forces exerted by charges are linear." (they add and scalar multiply linearly)

$$\nabla\cdot(E_1+E_2+\cdots+E_n) = \frac{\rho_1+\rho_2+\cdots+\rho_n}{\varepsilon_0}$$

For a collection of charges it holds that:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\sum q_i \frac{\boldsymbol{r}-\boldsymbol{r_i}}{\|\boldsymbol{r}-\boldsymbol{r_i}\|^3}$$

For a continuous charge distribution:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{r_i})\frac{\boldsymbol{r}-\boldsymbol{r_i}}{\|\boldsymbol{r}-\boldsymbol{r_i}\|^3}d\tau_i$$

With differentiation, the potential is:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r_i})}{\|\boldsymbol{r}-\boldsymbol{r_i}\|}d\tau_i$$

The Law of Cosines is often useful:

$$a^2+b^2-2ab\cos(\theta) = c^2$$

Electric Potential, Work

Electric potential energy is defined wrt an origin, and the change in energy is:

$$\Delta V = -\int_a^b \boldsymbol{E}\cdot d\boldsymbol{l}$$

Electric field comes from the potential:

$$E = -\nabla V \implies \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Point charge: $V = q/(4\pi\varepsilon_0\, r)$
Infinite line: $V = -\lambda/(2\pi\varepsilon_0)\cdot\ln(s/\alpha)$
Infinite sheet: $V = \sigma(z-\alpha)/(2\varepsilon_0)$
Sphere shell: $V = \sigma a^2/(\varepsilon_0 r)$

Work is the energy of a path of force:

$$W = -\int_\infty^r \boldsymbol{F}\cdot d\boldsymbol{l} = -q\Delta V$$

Work for a discrete distribution is:

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{\text{pairs}}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum q_i V(r_i)$$

Work for a continuous distribution is:

$$W = \frac{1}{2}\int \rho V d\tau \xrightarrow{\text{all space}} W = \frac{\varepsilon_0}{2}\int E^2 d\tau$$

Conductors & Capacitors

In conductors, charge accumulates on the outer surface(s) so that there is no net electric field within the conductor.

In a capacitor, capacitance is given as:

$$C \equiv \frac{q}{\Delta V}$$

The work invested in a capacitor is:

$$W = \frac{C\Delta V^2}{2}$$

Polarization & Multipoles

The Law of Cosines combined with Taylor Expansion, for $\boldsymbol{r}$ from origin to the point, and $\boldsymbol{r'}$ from origin to dq:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos(\alpha))$$

Now, with dipole moment $\boldsymbol{p} \equiv q\boldsymbol{d}$, V may be written as a series where Coulomb's Law is the potential for a monopole term. For the dipole term:

$$V_d = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{r}}\cdot\boldsymbol{p}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos(\theta)}{r^2}$$

More generally, the potential is a series:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^n}\int (r')^n P_n(\cos(\alpha))\rho(r')\, d\tau'$$

The polarization is $\boldsymbol{P} \equiv \boldsymbol{p}/V$, noting that in general, $\boldsymbol{P} \equiv (\int \boldsymbol{r'}\rho(\boldsymbol{r'})\, d\tau')/V$. Bound charges are thus determined:

$$\rho_b = -\nabla\cdot\boldsymbol{P}$$

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}}$$

From these the potential is found as:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\oint_S \frac{\sigma_b}{|\boldsymbol{r}-\boldsymbol{r'}|}da' + \int_V \frac{\rho_b}{|\boldsymbol{r}-\boldsymbol{r'}|}d\tau'\right)$$

With $\boldsymbol{P} = (\varepsilon-\varepsilon_0)\boldsymbol{E}$, displacement is:

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$$

This allows an analog to Gauss's Law:

$$\oint_S \boldsymbol{D}\cdot d\boldsymbol{a} = q_f$$

Method of Images

The method of images sometimes works to solve the Laplace Equation:

- Mirror position and charge
- Find V by Coulomb's Law
- Find $\boldsymbol{E}$ with $\boldsymbol{E} = -\nabla V$
- Find W by integration, division

Legendre Polynomials

The first few Legendre Polynomials:

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = \tfrac{1}{2}(3x^2-1)$$

$$P_3 = \tfrac{1}{2}(5x^3-3x)$$

$$P_4 = \tfrac{1}{8}(35x^4-30x^2+3)$$

$$P_5 = \tfrac{1}{8}(63x^5-70x^3+15x)$$

Generally, Rodrigues' formula is:

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Rectangular Laplace Eqn.

The method of separation of variables is often useful to solve the Laplace Equation. In rectangular coordinates, a general solution for infinite z is:

$V = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$

Method to solve rectangular problems:

- Use ansatz above (or similar)
- Remove coefficients with B.Cs.
- Solve for remaining coefficient
  - ⋆ Fourier's trick may help

Fourier's Trick is the multiplication on both sides by $\sin(n'\pi y/a)$ followed by integration where orthogonality then reduces the problem to something like:

$$c_n = \frac{2}{a}\int_0^a dy\ V_0(y)\sin\left(\frac{n\pi y}{a}\right)$$

Spherical Laplace Eqn.

Separation of variables also works in spherical geometries. Here, the general solution with azimuthal symmetry is:

$$V = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

Method to solve spherical problems:

- Use ansatz above (or similar)
- Remove coefficients with B.Cs.
- Solve for remaining coefficient
  - ⋆ break V into P_l and solve
  - ⋆ (or) break $E = -\nabla V$ into P_l
  - ⋆ (or) break σ up with boost

A useful orthogonality relation is:

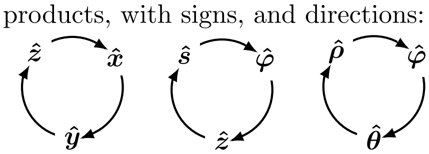
$$\int_0^{\pi} d\theta\ P_l P_{l'}\sin(\theta) = \frac{2}{2l+1}\delta_{ll'}$$

For finding the electric field this helps:

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}}$$

Geometry

Cyclic permutations summarize cross products, with signs, and directions:



If a vector direction is fixed, no prime, if it changes, like $\hat{\boldsymbol{s}}$, then add a prime.

Currents and Forces

$$\boldsymbol{I} = \lambda\boldsymbol{v}, \qquad \boldsymbol{F}_{\text{mag}} = \int(\boldsymbol{I}\times\boldsymbol{B})dl$$

$$\boldsymbol{K} = \sigma\boldsymbol{v}, \qquad \boldsymbol{F}_{\text{mag}} = \int(\boldsymbol{K}\times\boldsymbol{B})da$$

$$\boldsymbol{J} = \rho\boldsymbol{v}, \qquad \boldsymbol{F}_{\text{mag}} = \int(\boldsymbol{J}\times\boldsymbol{B})d\tau$$

Biot-Savart, Ampère's Law

Biot-Savart Law for steady current, $\partial_t\rho = 0$ which also holds for $\boldsymbol{K}$ and $\boldsymbol{J}$:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{I}\times(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3}dl'$$

Schema for solving B.S. Law problems:

- Note: $d\boldsymbol{l}$ and $\boldsymbol{I}$ are often collinear
- Find $\boldsymbol{r}$, $\boldsymbol{r}'$, and $\boldsymbol{I}$ (or $d\boldsymbol{l}$)
- Compute $d\boldsymbol{l}\times\boldsymbol{r}$, and $d\boldsymbol{l}\times\boldsymbol{r}'$
- Break up and evaluate integral

Ampère's Law works best when $\boldsymbol{B} \parallel d\boldsymbol{l}$, and behaves basically like Gauss's Law:

$$\oint\boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{\text{enc}}$$

The Vector Potential

The vector potential is found with:

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(r)d\tau'}{\|\boldsymbol{r}-\boldsymbol{r}'\|}$$

Some nice properties of the $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\nabla\times\boldsymbol{A} = \boldsymbol{B}; \qquad \nabla^2\boldsymbol{A} = -\mu_0 J$$

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}; \qquad \nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\cdot\boldsymbol{J} = -\partial\rho/\partial t$$

Multipoles, Magnetization

The vector potential may be expanded as a multipole, in particular the dipole:

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos(\theta)dl = -\frac{\mu_0 I}{4\pi r^2}\hat{\boldsymbol{r}}\times\oint da' = \frac{\mu_0 I}{4\pi r^2}\boldsymbol{m}\times\hat{\boldsymbol{r}}$$

Magnetic dipole moment $\boldsymbol{m} \equiv I\int da'\hat{\boldsymbol{n}}$. Magnetization is $\boldsymbol{M} = \boldsymbol{m}/V$, so thence:

$$\boldsymbol{J}_b = \nabla\times\boldsymbol{M}; \qquad \boldsymbol{K}_b = \boldsymbol{M}\times\hat{\boldsymbol{n}}$$

From this the vector potential is then:

$$V = \frac{\mu_0}{4\pi}\left(\oint_S\frac{\boldsymbol{K}_b}{|\boldsymbol{r}-\boldsymbol{r}'|}da' + \int_V\frac{\boldsymbol{J}_b}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau'\right)$$

Define the magnetic auxiliary field as:

$$\boldsymbol{H} \equiv \boldsymbol{B}/\mu_0 - \boldsymbol{M} = \mu\boldsymbol{B}$$

For the purpose of finding free current:

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f$$

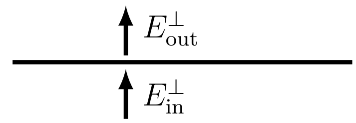
Another Gauss's Law analog is:

$$\oint\boldsymbol{H}\cdot dl = I_{f,\text{enc}}$$

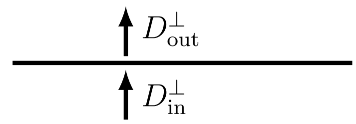
Scalar potential if $J_f = 0$ and $\nabla\times\boldsymbol{M} = 0$:

$$V \equiv -\int\boldsymbol{H}\cdot d\boldsymbol{l} \implies \boldsymbol{H} = -\nabla V$$

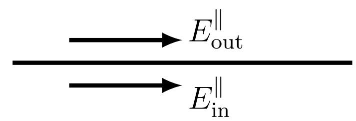
Boundary Conditions



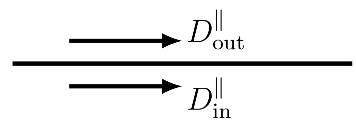
$$E^{\perp}_{\text{out}} - E^{\perp}_{\text{in}} = \sigma/\varepsilon_0$$



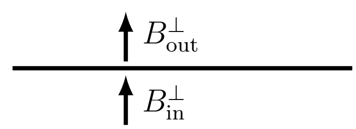
$$D^{\perp}_{\text{out}} - D^{\perp}_{\text{in}} = \sigma_f$$



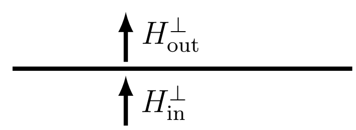
$$E^{\parallel}_{\text{out}} - E^{\parallel}_{\text{in}} = 0$$



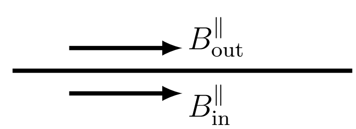
$$D^{\parallel}_{\text{out}} - D^{\parallel}_{\text{in}} = P^{\parallel}_{\text{out}} - P^{\parallel}_{\text{in}}$$



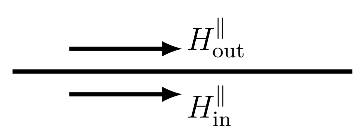
$$B^{\perp}_{\text{out}} - B^{\perp}_{\text{in}} = 0$$



$$H^{\perp}_{\text{out}} - H^{\perp}_{\text{in}} = M^{\perp}_{\text{in}} - M^{\perp}_{\text{out}}$$



$$B^{\parallel}_{\text{out}} - B^{\parallel}_{\text{in}} = \mu_0 K\times\hat{\boldsymbol{n}}$$



$$H^{\parallel}_{\text{out}} - H^{\parallel}_{\text{in}} = K_f\times\hat{\boldsymbol{n}}$$