Solving Differential Eqs

Begin with a linear, second order ODE:

$$\ddot{x}(t) + \alpha\,\dot{x}(t) + \beta\,x(t) = y(t)$$

Now note that the solution to this is:

$$x(t) = x_h(t) + x_p(t)$$

Find homogeneous solution via Ansatz:

$$x_h(t) = e^{\lambda t} \implies \lambda^2 + \alpha\lambda + \beta = 0$$

Find particular solution via Ansatz:

$$x_p(t) = \begin{cases} \text{exp} & c_1 \exp(\omega t) \\ \text{trig} & c_1\cos(\omega t) + c_2\sin(\omega t) \\ \text{poly} & \sum_{i=0}^{n} c_i t^i \end{cases}$$

Then solve the algebraic equations and combine solutions for general solution.

Decoupling ODE Systems

General method to decouple equations:

- Write your system of equations
- Differentiate one and rearrange
- Substitute into other, simplify
- Repeat as needed until decoupled

Vector Identities

The dot product is given by:

$$\boldsymbol{v}\cdot\boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos(\theta)$$

$$\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{v_{\hat{\imath}}w_{\hat{\imath}}} + \boldsymbol{v_{\hat{\jmath}}w_{\hat{\jmath}}} + \boldsymbol{v_{\hat{k}}w_{\hat{k}}}$$

The cross product is given by:

$$\|\boldsymbol{v}\times\boldsymbol{w}\| = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\sin(\theta)$$

$$\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \boldsymbol{v_{\hat{\imath}}} & \boldsymbol{v_{\hat{\jmath}}} & \boldsymbol{v_{\hat{k}}} \\ \boldsymbol{w_{\hat{\imath}}} & \boldsymbol{w_{\hat{\jmath}}} & \boldsymbol{w_{\hat{k}}} \end{vmatrix}$$

Scalar and Vector Triple Products:

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

Scalar and Vector Quadruple Products:

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\cdot\boldsymbol{d})$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{d})\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{d}$$

Other Coordinate Systems

Rectangular, $x_1 = x_1;\ x_2 = x_2;\ x_3 = x_3$:

$$ds^2 = dx^2 + dy^2 + dz^2;\ dv = dx\,dy\,dz$$

Cylindrical:

$$x_1 = r\cos(\phi);\ x_r = r\sin(\phi);\ x_3 = z$$

$$r = \sqrt{x_1^2 + x_2^3};\ \phi = \tan^{-1}(x_2/x_1);\ z = x_3$$

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2;\ dv = r\ dr\,d\phi\,dz$$

Spherical:

$$x_1 = r\sin(\theta)\cos(\phi);\ x_2 = r\sin(\theta)\sin(\phi);\ x_3 = r\cos(\theta)$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2};\ \theta = \cos^{-1}(x_3/r);\ \phi = \tan^{-1}(x_2/x_1)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2;\ dv = r^2\sin(\theta)\ dr\,d\theta\,d\phi$$

Trigonometric Identities

Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

$$\sin(\theta) = (e^{i\theta} - e^{-i\theta})/2i;\ \cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$$

The Pythagorean identity is:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

Sum and difference identities are:

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Fourier sum to product identities are:

$$\sin(mx)\cos(nx) = \tfrac{1}{2}[\sin((m+n)x) + \sin((m-n)x)]$$

$$\cos(mx)\cos(nx) = \tfrac{1}{2}[\cos((m+n)x) + \cos((m-n)x)]$$

$$\sin(mx)\sin(nx) = \tfrac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]$$

Hamilton's Principle

Lagrangian, and generalized momenta:

$$L = T - U; \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

The Euler-Lagrange Equations are:

$$\delta\int_{t_1}^{t_2} dt\,L = 0 \implies \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

The Hamiltonian is given by:

$$H = \left(\sum p_j\dot{q}_j\right) - L$$

If generalized coordinates/constraints are time independent, and potential is independent of $\dot{q}$, then $H = T + U$.

Linear and angular momentum are:

$$\boldsymbol{p_{\text{lin}}} = m\dot{\boldsymbol{x}}_{\boldsymbol{i}}; \qquad \boldsymbol{p_{\text{ang}}} = \boldsymbol{r}\times\boldsymbol{p}$$

Hamilton's equations are given by:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

The Liouville Theorem says that the canonical volume is constant through a motion, $d\rho/dt = 0$:

$$N = \rho dV; \quad dV = dq_1 \dots dq_s dp_1 \dots dp_s$$

Obtaining Hamilton's Eqs

- Draw a picture of your system
- Find the Lagrangian
  - find x, y in generalized form
  - differentiate x and y
  - substitute for T and U
- Find Hamilton's Equations
  - Find generalized momenta
  - Hamiltonian as a sum
  - Convert $\dot{x}$ to terms of p_x
  - Find Hamilton's Equations

Central Force Motion

For central force motion:

$$F = -\frac{k}{r^2}; \qquad U = -\int dr\,F(r) = -\frac{k}{r}$$

Lagrangian with $\mu = (m_1 m_2)/(m_1 + m_2)$:

$$L = \frac{1}{2}\mu|\dot{r}|^2 - U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

The angular momentum and energy:

$$\ell = \mu r^2\dot{\theta}; \qquad U_\ell = \frac{\ell^2}{2\mu r^2}$$

The total energy is found to be:

$$E = \frac{1}{2}\mu\dot{r}^2 + U_\ell + U \iff \dot{r} = \pm\sqrt{\frac{2}{\mu}(E - U - U_\ell)}$$

Dynamics of Systems

The center of mass is, with $dm = \rho\,dV$:

$$\boldsymbol{R_{\text{cm}}} = \sum\frac{m_i}{M}\boldsymbol{r_i}; \quad \boldsymbol{R_{\text{cm}}} = \frac{1}{M}\int \boldsymbol{r}\,dm$$

Linear momentum and acceleration:

$$\boldsymbol{P} = M\dot{\boldsymbol{R}}_{\text{cm}}; \qquad \boldsymbol{F} = \dot{\boldsymbol{P}} = M\ddot{\boldsymbol{R}}_{\text{cm}}$$

The angular momentum is:

$$\boldsymbol{L} = \boldsymbol{R_{\text{cm}}}\times\boldsymbol{P} + \sum \boldsymbol{r_i}\times\boldsymbol{p_i}$$

The Kinetic Energy is:

$$T = T_{\text{trans}} + T_{\text{rot}} = \tfrac{1}{2}MV^2 + \sum m_i v_i^2$$

Momentum is conserved, ex. rockets:

$$v = v_0 + v_{\text{ex}}\ln(m_0/m)$$

Solving Elastic Collisions

- Draw a picture of your system
- Write conservation of $\boldsymbol{p}$ and E
- Solve for easily found quantities
  - With scattering angle ψ, $\cos^2(\psi) + \sin^2(\psi) = 1$
  - Isolate variables from others
- Find the quantities that you want
  - Substitute solved variables
  - Put everything as knowns
  - Algebra to completion

Scattering Cross Sections

The differential cross section is:

$$\sigma_{\text{diff}}(\theta) = \frac{d\sigma_{\text{tot}}}{d\Omega} = \frac{b}{\sin(\theta)}\left|\frac{db}{d\theta}\right|$$

The total cross section is:

$$\sigma_{\text{tot}} = \int\frac{d\sigma_{\text{tot}}}{d\Omega}d\Omega = \int\sigma_{\text{diff}}(\theta)2\pi\sin(\theta)d\theta$$

Scattering angle; angular momentum $\ell = b\sqrt{2\mu T_{r=\infty}}$, $E = T_{r=\infty}$, $\theta = \pi - 2\Theta$:

$$\Delta\Theta = \int_{r_{\text{min}}}^{r_{\text{max}}}\frac{\ell/r^2\ dr}{\sqrt{2\mu(E - U - U_\ell)}} \Rightarrow \Theta = \int_{r_{\text{min}}}^{\infty}\frac{b/r^2\ dr}{\sqrt{1 - (b^2/r^2) - (U/T_\infty)}}$$

Rutherford Scattering has infinite σ_{tot}:

$$U(r) = \frac{k}{r};\ \sigma_{\text{diff}}(\theta) = \frac{k^2}{(4\,T_{r=\infty})^2}\cdot\frac{1}{\sin^4(\theta/2)}$$

Non-Inertial Frames

For any $\boldsymbol{Q}$ in rotating coordinates:

$$\left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{Q}$$

Including all fictitious forces, one finds:

$$\boldsymbol{F}_{\text{eff}} = \boldsymbol{F} - m\ddot{\boldsymbol{R}} - m(\dot{\boldsymbol{\omega}}\times\boldsymbol{r}) - m(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - 2m(\boldsymbol{\omega}\times\boldsymbol{v_r})$$

On Earth $\dot{\boldsymbol{\omega}} = 0$, and, $m\ddot{\boldsymbol{R}} = m(\dot{\boldsymbol{\omega}}\times\boldsymbol{r})$:

$$\boldsymbol{F}_{\text{eff}} = \boldsymbol{F}_{\text{other ext}} + m\boldsymbol{g} - 2m(\boldsymbol{\omega}\times\boldsymbol{v_r})$$

Rigid Body Dynamics

Moment of inertia tensor components:

$$I_{ij} \equiv \sum m_\alpha\left(\left(\delta_{i,j}\sum_k^{1,2,3} x_{\alpha,k}^2\right) - x_{\alpha,i}x_{\alpha,j}\right)$$

$$I_{ij} = \int_v dv\ \rho(r)\left(\left(\delta_{i,j}\sum x_k^2\right) - x_i x_j\right)$$

Moving the origin by $\boldsymbol{a}$ from $\widetilde{\boldsymbol{r}}$ to $\boldsymbol{r}_{\text{cm}}$:

$$I_{ij} = \widetilde{I}_{ij} - M(a^2\delta_{ij} - a_i a_j)$$

Angular momentum may be computed:

$$L_i = \sum I_{ij}\omega_j \implies \boldsymbol{L} = \boldsymbol{I}\cdot\boldsymbol{\omega}$$

Rotational kinetic energy is found as:

$$T_{\text{rot}} = \tfrac{1}{2}\sum m_i(\boldsymbol{\omega}\times\boldsymbol{r_\alpha})^2 = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$$

$$= \tfrac{1}{2}\sum I_{ij}\omega_i\omega_j \quad \text{sum over } i \text{ and } j$$

Principal moments of inertia are the eigenvalues and principal axes are the eigenvectors of the inertia tensor. (It is easier if already partially diagonalized)

Euler Angles

Euler angles quantify angular velocity with angles from fixed to body frames:

$$\omega_1 = \dot\phi_1 + \dot\theta_1 + \dot\psi_1 = \dot\phi\sin(\theta)\sin(\psi) + \dot\theta\cos(\psi)$$

$$\omega_2 = \dot\phi_2 + \dot\theta_2 + \dot\psi_2 = \dot\phi\sin(\theta)\cos(\psi) - \dot\theta\sin(\psi)$$

$$\omega_3 = \dot\phi_3 + \dot\theta_3 + \dot\psi_3 = \dot\phi\cos(\theta) + \dot\psi$$

Torque is the sum of tangential forces:

$$\boldsymbol{\Gamma} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{\text{f}} = \left(\frac{\partial\boldsymbol{L}}{\partial t}\right)_{\text{b}} + \boldsymbol{\omega}\times\boldsymbol{L} \Rightarrow \left(\frac{\partial\boldsymbol{L}}{\partial t}\right)_{\text{b}} = \boldsymbol{\Gamma} - \boldsymbol{\omega}\times\boldsymbol{L}$$

For motions with external forces Γ_i:

$$I_1\dot\omega_1 = \Gamma_1 - \omega_2\omega_3(I_3 - I_2)$$

$$I_2\dot\omega_2 = \Gamma_2 - \omega_1\omega_3(I_1 - I_3)$$

$$I_3\dot\omega_3 = \Gamma_3 - \omega_1\omega_2(I_2 - I_1)$$

Decoupling ODE Systems

Define normal coordinates as:

$$\boldsymbol{Q_i} = \boldsymbol{q}\cdot\boldsymbol{v_i}; \quad \boldsymbol{v_i} \text{ normalized}$$

Equations decouple with substitution

Ex. $\boldsymbol{v_1} = \frac{1}{\sqrt{2}}\binom{1}{1}$, and $\boldsymbol{v_2} = \frac{1}{\sqrt{2}}\binom{1}{-1}$:

$$Q_1 \equiv (q_1 + q_2)/2; \quad Q_2 \equiv (q_1 - q_2)/2$$

Coupled Oscillations

With coupled oscillations of two or more oscillators, there appear normal modes $\boldsymbol{v_i}$, and normal frequencies ω_i, which may be evaluated with initial conditions to find a general solution.

- Draw a picture of your system
- Find L (as in Hamilton's Eqs)
- Write Euler-Lagrange Equations
  - Make it only include $\ddot{q}_i$ & q_i
- Differentiate and use Ansatz
  $q_1 = A_1 e^{i\omega t};\ \ldots\ q_n = A_n e^{i\omega t}$
- Divide by $e^{i\omega t}$
- Write matrix equation $\boldsymbol{MA} = \boldsymbol{0}$
- Solve secular eqn $|\boldsymbol{M}| = 0$ for ω_i^2
- Write the n. mode frequencies ω_i
- Solve for $\boldsymbol{v_i}$ if $(\boldsymbol{M} - \omega_i^2\boldsymbol{1})\boldsymbol{v_i} = \boldsymbol{0}$
  - Explicitly write the vectors
  - Can normalize $\boldsymbol{v_i^T M v_i} = 1$
  - $\boldsymbol{v_i}$ are orthogonal if ω_i^2 are distinct, for symmetric $\boldsymbol{M}$
- Write the general solution:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \sum_i P_i\boldsymbol{v_i}\cos(\omega_i t - \phi_i)$$

$$\binom{1}{1}, \binom{1}{-1} \Rightarrow \begin{pmatrix} A\cos(\omega_1 t - \alpha) + B\cos(\omega_2 t - \beta) \\ A\cos(\omega_1 t - \alpha) - B\cos(\omega_2 t - \beta) \end{pmatrix}$$

- Solve for boundary conditions
  - Explicitly write derivatives

Weak Coupling

The goal is to find $q_1(t)$ and $q_2(t)$.

- Write in uncoupled frequencies
  - $\omega_0 = (\omega_1 + \omega_2)/2$
  - $\implies \omega_1 \approx \omega_0(1+\epsilon)$
  - $\implies \omega_2 \approx \omega_0(1-\epsilon)$
- Solve for the general solution
- Make the solution a product
  - Use Fourier's Identities
  - Write in terms of ϵ

Loaded String

The Lagrangian for the loaded string with n masses m of spacing d at $q_j(t)$:

$$L = \frac{1}{2}\sum_{j=1}^{n+1} m\dot{q}_j^2 - \frac{\tau}{d}(q_{j-1} - q_j)^2$$

Which has the equations of motion:

$$\ddot{q}_j = \frac{\tau}{md}(q_{j-1} - 2q_j + q_{j+1})$$

With the Ansatz $q_j(t) = a_j e^{i\omega t}$, then $q_j(t)$ is sum over $r = 1, \ldots, n$ frequencies:

$$a_{jr} = a_r\sin\left(j\frac{r\pi}{n+1}\right);\ \omega_r = \sqrt{\frac{4\tau}{md}}\sin\left(\frac{r\pi}{2(n+1)}\right)$$

Continuous Systems

General formulae for oscillating strings: $v = \sqrt{T/\rho}$, $\lambda f = v$, $\omega = 2\pi f$, $k = \omega/v = 2\pi/\lambda$

If both ends are at rest, Fourier Sines:

$$a_n = \frac{2}{L}\int_0^L dx\ q(x,0)\sin\left(\frac{n\pi}{L}x\right); b_n \sim \dot{q}(x,0)$$

$$\omega_n = \sqrt{\frac{\tau}{\rho}}\frac{2}{d}\sin\left(\frac{n\pi}{L}\frac{d}{2}\right) \xRightarrow{d\to 0} \omega_n = \sqrt{\frac{\tau}{\rho}}\frac{n\pi}{L}$$

$$q(x,t) = \sum_n\left(a_n\cos(\omega_n t) + \frac{b_n}{\omega_n}\sin(\omega_n t)\right)\sin\left(\frac{n\pi}{L}x\right)$$

For driving forces, think about how to expand $F(x,t)$, and how the Fourier Expansion changes. Draw a diagram.

Wave equation, Helmholtz equation:

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}; \quad \frac{\partial^2\psi_n}{\partial x^2} + k_n^2\psi = 0$$

May be solved by two separation types:

$$\Psi(x,t) = f(x+vt) + g(x-vt)$$

$$\text{(or)}\quad \Psi(x,t) = \sum_n \psi_n(x)\exp(i\omega_n t)$$

Phase, v_p, and group, v_g, velocities:

$$v_p = \frac{\omega}{k} \text{ and } v_g = \frac{d\omega}{dk} \text{ for } k = \frac{n\pi}{L}$$

Special Relativity

Einstein's Special Relativity postulates:

I. Same physical laws in all inertial frames.
II. Light's speed is a universal constant.

Times and distances measured in their own frames are known as "proper."

Galilean invariance gives the transform between frames of $x_1' = x_1 - vt$, $t' = t$. With v in the x_1 direction, the Lorentz transformation from the stationary x_1 frame to the moving x_1' frame is, with $\beta = v/c$ and the factor $\gamma = 1/\sqrt{1-\beta^2}$:

$$\begin{Bmatrix} x_1' = \gamma(x_1 - vt) \\ x_2' = x_2 \\ x_3' = x_3 \\ t' = \gamma\left(t - \frac{vx_1}{c^2}\right) \end{Bmatrix} \begin{Bmatrix} x_1 = \gamma(x_1' + vt') \\ x_2 = x_2' \\ x_3 = x_3' \\ t = \gamma\left(t' + \frac{vx_1'}{c^2}\right) \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{x}_1' = (\dot{x}_1 - v)/(1 - \dot{x}_1 v/c^2) \\ \dot{x}_2' = \dot{x}_2/(\gamma(1 - \dot{x}_1 v/c^2)) \\ \dot{x}_3' = \dot{x}_3/(\gamma(1 - \dot{x}_1 v/c^2)) \end{Bmatrix}$$

With l in x frame, and observer in x' frame seeing both ends at same time:

$$l' = l/\gamma = l\sqrt{1-\beta^2}$$

With t at a fixed location in x frame, and observer in x' frame:

$$\Delta t' = t_2' - t_1' = \gamma\Delta t = \Delta t/\sqrt{1-\beta^2}$$

Relativistic Doppler effect for receding $v < 0$ and nearing $v > 0$, with $\lambda f = c$:

$$f_{\text{detect}} = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}f_{\text{emit}} \Leftrightarrow \lambda_{\text{d}} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\lambda_{\text{e}}$$