# Physics 105A at UCLA ◊ Formula Sheet (1 of 2)

## Jacobians

The Jacobian matrix in 2D is

$$J = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix}$$

Whence the integral is ($|J| = \det(J)$):

$$\int dxdy\, f(x,y) = \int |J| dx'dy'\, f(x(x',y'),...)$$

Commonly encountered Jacobians:

$$|J|_{\text{cartesian}} = 1$$
$$|J|_{\text{polar}} = r$$
$$|J|_{\text{cylindrical}} = r$$
$$|J|_{\text{spherical}} = r^2 \sin(\theta)$$

## Path Lengths

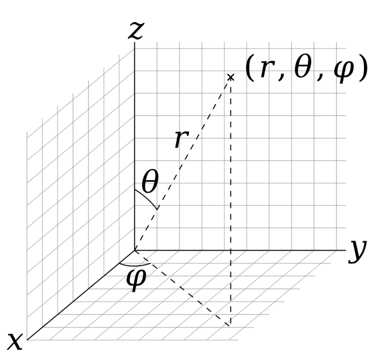
The differential path lengths are

$$ds_{\text{cartesian}} = \sqrt{dx^2 + dy^2 + dz^2}$$
$$ds_{\text{polar}} = \sqrt{dr^2 + r^2 d\phi^2}$$
$$ds_{\text{cylindrical}} = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}$$
$$ds_{\text{sphere}} = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

## Spherical Coordinates

Now in spherical coordinates we have

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$




## Polar Forces

We can decompose forces as

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$
$$F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

This identity sometimes is helpful:

$$\dot{r}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

## Trigonometry

The law of cosines is:

$$a^2 + b^2 - 2ab\cos(\gamma) = c^2$$

The Pythagorean identities are:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\sec^2(\theta) - \tan^2(\theta) = 1$$
$$\csc^2(\theta) - \cot^2(\theta) = 1$$

The double-angle identities are:

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

## General Math

The dot product of two vectors:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \|\boldsymbol{A}\|\|\boldsymbol{B}\|\cos(\theta)$$

The cross product of two vectors:

$$\|\boldsymbol{A} \times \boldsymbol{B}\| = \|\boldsymbol{A}\|\|\boldsymbol{B}\|\sin(\theta)$$

The divergence theorem:

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{A}) dV = \oint_{\partial V} \boldsymbol{A} \cdot d\boldsymbol{S}$$

Stokes theorem is:

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{l}$$

## Physics 1 Knowledge

The center of mass position is:

$$\boldsymbol{r}_{\text{com}} = \frac{1}{m_{\text{tot}}} \sum_i m_i \boldsymbol{r}_i$$

And the linear momentum is:

$$\boldsymbol{p} = m_{\text{tot}} \dot{\boldsymbol{r}}_{\text{com}}$$

Where Newton's second law is:

$$\dot{\boldsymbol{p}} = \boldsymbol{F} = m\boldsymbol{a}$$

The kinetic energy is then:

$$T = \frac{p^2}{2m}$$

And the total energy is

$$E = T + U$$

## More Physics 1 Knowledge

For conservative forces

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0$$

which holds if and only if

$$\boldsymbol{F} = -\boldsymbol{\nabla} U$$

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We also have the work-energy theorem

$$dT = \boldsymbol{F} \cdot d\boldsymbol{r}$$

which is

$$W = -\int \boldsymbol{F} \cdot d\boldsymbol{r}$$

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The position as a function of time is:

$$x(t) = x(0) + \int_0^t d\tau\ v_x(\tau)$$

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The time-average of a quantity $f$ is

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau dt\ f(t)$$

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For a spring-block system with spring stiffness $k$, the motion is

$$x(t) = A\cos(\omega t - \phi)$$

$\omega = \sqrt{k/m}$ is the angular frequency.

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Projectile motion with only gravity above a flat surface lasts for a time

$$t_{\text{final}} = \frac{2v_0 \sin(\theta)}{g}$$

Where the distance traveled is

$$x(t_{\text{final}}) = \frac{v_0^2}{g}\sin(2\theta)$$

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Non-relativistic rocket motion is described by the change in momentum with respect to time

$$\dot{p} = m\dot{\boldsymbol{v}} = -\dot{m}\boldsymbol{v}_{\text{exhaust}} + \boldsymbol{F}_{\text{ext}}$$

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The motion of a massive particle in the presence of a magnetic field is circular with "cyclotron" radius

$$r = \frac{v}{\omega}$$

where the cyclotron frequency is

$$\omega = \frac{qB}{m}$$

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Elastic collisions are ones for which $T$ is preserved. Additionally when no external forces are applied, $\boldsymbol{p}$, $\boldsymbol{L}$, and $E$ are conserved.

## Drag & Terminal Velocity

At small $v$, drag is linear for a sphere of diameter $d$

$$\boldsymbol{F}_{\text{drag}} = -3\pi\eta d\boldsymbol{v}$$

At large $v$, drag is quadratic

$$\boldsymbol{F}_{\text{drag}} = -\gamma d^2 v^2 \hat{\boldsymbol{v}}$$

At terminal velocity, the forces balance

$$v_{\text{ter}}^{\text{lin}} = \frac{mg}{b}$$

And for quadratic drag

$$v_{\text{ter}}^{\text{quad}} = \sqrt{\frac{mg}{c}}$$

## Driven & Damped Oscillations

Generically a driven-damped oscillator takes the form

$$\ddot{x} + 2\beta\dot{x} + \omega_0 x = f(t)$$

Which is solved by

$$x(t) = x_h + x_p$$

$x_h$ is the homogeneous solution that can be found from the characteristic polynomial, and $x_p$ is the particular solution that can be found by the method of undetermined coefficients.

When the driving vanishes

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

and the solution is

$$x(t) = e^{-\beta t}(c_1 e^{\sqrt{\beta^2-\omega_0^2}t} + c_2 e^{-\sqrt{\beta^2-\omega_0^2}t})$$

There are then four regimes:

- $\beta = 0$: undamped
- $\beta < \omega_0$: underdamped
- $\beta = \omega_0$: critically damped
- $\beta > \omega_0$: overdamped

## Angular Momentum

The angular momentum is ($\omega = \dot{\phi}$)

$$\boldsymbol{L} = I\boldsymbol{\omega} = \boldsymbol{r} \times \boldsymbol{p}$$

For example, the $z$-component is

$$L_z = xp_y - yp_x$$

The torque is

$$\boldsymbol{\Gamma} = \dot{\boldsymbol{L}} = \boldsymbol{r} \times \boldsymbol{F}$$

## Moment of Inertia

The moment of inertia is

$$I = \sum_i m_i r_i^2$$

Which can be obtained from the com frame using the parallel axis theorem:

$$I = I_{\text{com}} + md^2$$

For a uniform disk:

$$I_{\text{disk}} = \frac{1}{2}mr^2$$

For a uniform sphere:

$$I_{\text{sphere}} = \frac{2}{5}mr^2$$

The angular kinetic energy is then

$$T = \frac{1}{2}I\omega^2$$

## Reduced Mass & Effective Potential

The reduced mass $\mu$ of a two-body system is defined so that we obtain

$$\boldsymbol{L} = \boldsymbol{r} \times \mu\dot{\boldsymbol{r}}$$

For radial motion there is the analogue of Newton's second law

$$\mu\ddot{r} = -\frac{d}{dr}U_{\text{eff}}(r)$$

Where the effective potential is

$$U_{\text{eff}}(r) = U(r) + \frac{L^2}{2\mu r^2}$$

## Kepler's Laws

1. The planets orbit the sun in ellipses, and the sun is at one focus
2. Equal areas are swept out in equal times

$$\frac{d\boldsymbol{A}}{dt} = \frac{\boldsymbol{L}}{2m} = \frac{r^2\boldsymbol{\omega}}{2}$$

3. The period is proportional to the semi-major axis, $a$, to the 3/2

$$\tau = \sqrt{\frac{4\pi^2}{G_N M_\odot}}a^{3/2}$$

In polar coordinates the motion is

$$r(\phi) = \frac{c}{1 + \epsilon\cos(\phi)}$$

Where the eccentricity is

$$\epsilon = \frac{AL^2}{\gamma\mu}$$

## Euler-Lagrange Equations

The Lagrangian is

$$\mathcal{L} = T - U$$

Which leads to the Euler-Lagrange equations, which are equivalent to Newton's equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0$$

It turns out that the trajectories systems take are those that make the action $S$ stationary

$$S = \int dt\ \mathcal{L}$$